LONDON MATHEMATICAL SOCIETY LECTURE NOTE SERIES

Managing Editor: Professor J.W.S. Cassels, Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, 16 Mill Lane, Cambridge CB2 1SB, England

The titles below are available from booksellers, or, in case of difficulty, from Cambridge University Press.

34 Representation theory of Lie groups, M.F. ATIYAH *et al*
46 p-adic analysis: a short course on recent work, N. KOBLITZ
50 Commutator calculus and groups of homotopy classes, H.J. BAUES
59 Applicable differential geometry, M. CRAMPIN & F.A.E. PIRANI
66 Several complex variables and complex manifolds II, M.J. FIELD
69 Representation theory, I.M. GELFAND *et al*
76 Spectral theory of linear differential operators and comparison algebras, H.O. CORDES
77 Isolated singular points on complete intersections, E.J.N. LOOIJENGA
83 Homogeneous structures on Riemannian manifolds, F. TRICERRI & L. VANHECKE
86 Topological topics, I.M. JAMES (ed)
87 Surveys in set theory, A.R.D. MATHIAS (ed)
88 FPF ring theory, C. FAITH & S. PAGE
89 An F-space sampler, N.J. KALTON, N.T. PECK & J.W. ROBERTS
90 Polytopes and symmetry, S.A. ROBERTSON
92 Representation of rings over skew fields, A.H. SCHOFIELD
93 Aspects of topology, I.M. JAMES & E.H. KRONHEIMER (eds)
94 Representations of general linear groups, G.D. JAMES
95 Low-dimensional topology 1982, R.A. FENN (ed)
96 Diophantine equations over function fields, R.C. MASON
97 Varieties of constructive mathematics, D.S. BRIDGES & F. RICHMAN
98 Localization in Noetherian rings, A.V. JATEGAONKAR
99 Methods of differential geometry in algebraic topology, M. KAROUBI & C. LERUSTE
100 Stopping time techniques for analysts and probabilists, L. EGGHE
104 Elliptic structures on 3-manifolds, C.B. THOMAS
105 A local spectral theory for closed operators, I. ERDELYI & WANG SHENGWANG
107 Compactification of Siegel moduli schemes, C-L. CHAI
109 Diophantine analysis, J. LOXTON & A. VAN DER POORTEN (eds)
110 An introduction to surreal numbers, H. GONSHOR
113 Lectures on the asymptotic theory of ideals, D. REES
114 Lectures on Bochner-Riesz means, K.M. DAVIS & Y-C. CHANG
115 An introduction to independence for analysts, H.G. DALES & W.H. WOODIN
116 Representations of algebras, P.J. WEBB (ed)
118 Skew linear groups, M. SHIRVANI & B. WEHRFRITZ
119 Triangulated categories in the representation theory of finite-dimensional algebras, D. HAPPEL
121 Proceedings of *Groups - St Andrews 1985*, E. ROBERTSON & C. CAMPBELL (eds)
122 Non-classical continuum mechanics, R.J. KNOPS & A.A. LACEY (eds)
128 Descriptive set theory and the structure of sets of uniqueness, A.S. KECHRIS & A. LOUVEAU
129 The subgroup structure of the finite classical groups, P.B. KLEIDMAN & M.W. LIEBECK
130 Model theory and modules, M. PREST
131 Algebraic, extremal & metric combinatorics, M-M. DEZA, P. FRANKL & I.G. ROSENBERG (eds)
132 Whitehead groups of finite groups, ROBERT OLIVER
133 Linear algebraic monoids, MOHAN S. PUTCHA
134 Number theory and dynamical systems, M. DODSON & J. VICKERS (eds)
135 Operator algebras and applications, 1, D. EVANS & M. TAKESAKI (eds)
136 Operator algebras and applications, 2, D. EVANS & M. TAKESAKI (eds)
137 Analysis at Urbana, I, E. BERKSON, T. PECK, & J. UHL (eds)
138 Analysis at Urbana, II, E. BERKSON, T. PECK, & J. UHL (eds)
139 Advances in homotopy theory, S. SALAMON, B. STEER & W. SUTHERLAND (eds)
140 Geometric aspects of Banach spaces, E.M. PEINADOR & A. RODES (eds)
141 Surveys in combinatorics 1989, J. SIEMONS (ed)
144 Introduction to uniform spaces, I.M. JAMES
145 Homological questions in local algebra, JAN R. STROOKER
146 Cohen-Macaulay modules over Cohen-Macaulay rings, Y. YOSHINO
148 Helices and vector bundles, A.N. RUDAKOV *et al*
149 Solitons, nonlinear evolution equations and inverse scattering, M. ABLOWITZ & P. CLARKSON
150 Geometry of low-dimensional manifolds 1, S. DONALDSON & C.B. THOMAS (eds)
151 Geometry of low-dimensional manifolds 2, S. DONALDSON & C.B. THOMAS (eds)
152 Oligomorphic permutation groups, P. CAMERON
153 L-functions and arithmetic, J. COATES & M.J. TAYLOR (eds)
154 Number theory and cryptography, J. LOXTON (ed
155 Classification theories of polarized varieties, TAKAO FUJITA
156 Twistors in mathematics and physics, T.N. BAILEY & R.J. BASTON (eds)
158 Geometry of Banach spaces, P.F.X. MÜLLER & W. SCHACHERMAYER (eds)
159 Groups St Andrews 1989 volume 1, C.M. CAMPBELL & E.F. ROBERTSON (eds)
160 Groups St Andrews 1989 volume 2, C.M. CAMPBELL & E.F. ROBERTSON (eds)
161 Lectures on block theory, BURKHARD KÜLSHAMMER

162 Harmonic analysis and representation theory, A. FIGA-TALAMANCA & C. NEBBIA
163 Topics in varieties of group representations, S.M. VOVSI
164 Quasi-symmetric designs, M.S. SHRIKANDE & S.S. SANE
166 Surveys in combinatorics, 1991, A.D. KEEDWELL (ed)
167 Stochastic analysis, M.T. BARLOW & N.H. BINGHAM (eds)
168 Representations of algebras, H. TACHIKAWA & S. BRENNER (eds)
169 Boolean function complexity, M.S. PATERSON (ed)
170 Manifolds with singularities and the Adams-Novikov spectral sequence, B. BOTVINNIK
171 Squares, A.R. RAJWADE
172 Algebraic varieties, GEORGE R. KEMPF
173 Discrete groups and geometry, W.J. HARVEY & C. MACLACHLAN (eds)
174 Lectures on mechanics, J.E. MARSDEN
175 Adams memorial symposium on algebraic topology 1, N. RAY & G. WALKER (eds)
176 Adams memorial symposium on algebraic topology 2, N. RAY & G. WALKER (eds)
177 Applications of categories in computer science, M. FOURMAN, P. JOHNSTONE, & A. PITTS (eds)
178 Lower K- and L-theory, A. RANICKI
179 Complex projective geometry, G. ELLINGSRUD *et al*
180 Lectures on ergodic theory and Pesin theory on compact manifolds, M. POLLICOTT
181 Geometric group theory I, G.A. NIBLO & M.A. ROLLER (eds)
182 Geometric group theory II, G.A. NIBLO & M.A. ROLLER (eds)
183 Shintani zeta functions, A. YUKIE
184 Arithmetical functions, W. SCHWARZ & J. SPILKER
185 Representations of solvable groups, O. MANZ & T.R. WOLF
186 Complexity: knots, colourings and counting, D.J.A. WELSH
187 Surveys in combinatorics, 1993, K. WALKER (ed)
188 Local analysis for the odd order theorem, H. BENDER & G. GLAUBERMAN
189 Locally presentable and accessible categories, J. ADAMEK & J. ROSICKY
190 Polynomial invariants of finite groups, D.J. BENSON
191 Finite geometry and combinatorics, F. DE CLERCK *et al*
192 Symplectic geometry, D. SALAMON (ed)
193 Computer algebra and differential equations, E. TOURNIER (ed)
194 Independent random variables and rearrangement invariant spaces, M. BRAVERMAN
195 Arithmetic of blowup algebras, WOLMER VASCONCELOS
196 Microlocal analysis for differential operators, A. GRIGIS & J. SJÖSTRAND
197 Two-dimensional homotopy and combinatorial group theory, C. HOG-ANGELONI, W. METZLER & A.J. SIERADSKI (eds)
198 The algebraic characterization of geometric 4-manifolds, J.A. HILLMAN
199 Invariant potential theory in the unit ball of $\mathbf{C}^n$, MANFRED STOLL
200 The Grothendieck theory of dessins d'enfant, L. SCHNEPS (ed)
201 Singularities, JEAN-PAUL BRASSELET (ed)
202 The technique of pseudodifferential operators, H.O. CORDES
203 Hochschild cohomology of von Neumann algebras, A. SINCLAIR & R. SMITH
204 Combinatorial and geometric group theory, A.J. DUNCAN, N.D. GILBERT & J. HOWIE (eds)
205 Ergodic theory and its connections with harmonic analysis, K. PETERSEN & I. SALAMA (eds)
206 An introduction to noncommutative differential geometry and its physical applications, J. MADORE
207 Groups of Lie type and their geometries, W.M. KANTOR & L. DI MARTINO (eds)
208 Vector bundles in algebraic geometry, N.J. HITCHIN, P. NEWSTEAD & W.M. OXBURY (eds)
209 Arithmetic of diagonal hypersurfaces over finite fields, F.Q. GOUVÊA & N. YUI
210 Hilbert C^*-modules, E.C. LANCE
211 Groups 93 Galway / St Andrews I, C.M. CAMPBELL *et al*
212 Groups 93 Galway / St Andrews II, C.M. CAMPBELL *et al*
214 Generalised Euler-Jacobi inversion formula and asymptotics beyond all orders, V. KOWALENKO, N.E. FRANKEL, M.L. GLASSER & T. TAUCHER
215 Number theory 1992-93, S. DAVID (ed)
216 Stochastic partial differential equations, A. ETHERIDGE (ed)
217 Quadratic forms with applications to algebraic geometry and topology, A. PFISTER
218 Surveys in combinatorics, 1995, PETER ROWLINSON (ed)
220 Algebraic set theory, A. JOYAL & I. MOERDIJK
221 Harmonic approximation, S.J. GARDINER
222 Advances in linear logic, J.-Y. GIRARD, Y. LAFONT & L. REGNIER (eds)
223 Analytic semigroups and semilinear initial boundary value problems, KAZUAKI TAIRA
224 Computability, enumerability, unsolvability, S.B. COOPER, T.A. SLAMAN & S.S. WAINER (eds)
226 Novikov conjectures, index theorems and rigidity I, S. FERRY, A. RANICKI & J. ROSENBERG (eds)
227 Novikov conjectures, index theorems and rigidity II, S. FERRY, A. RANICKI & J. ROSENBERG (eds)
228 Ergodic theory of $\mathbf{Z}^d$ actions, M. POLLICOTT & K. SCHMIDT (eds)
229 Ergodicity for infinite dimensional systems, G. DA PRATO & J. ZABCZYK
230 Prolegomena to a middlebrow arithmetic of curves of genus 2, J.W.S. CASSELS & E.V. FLYNN
231 Semigroup theory and its applications, K.H. HOFMANN & M.W. MISLOVE (eds)
232 The descriptive set theory of Polish group actions, H. BECKER & A.S. KECHRIS
233 Finite fields and applications, S. COHEN & H. NIEDERREITER (eds)
234 Introduction to subfactors, V. JONES & V.S. SUNDER
235 Number theory 1993-94, S. DAVID (ed)

London Mathematical Society Lecture Note Series. 232

The Descriptive Set Theory of Polish Group Actions

Howard Becker
University of South Carolina

Alexander S. Kechris
California Institute of Technology

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSIGTY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521576055

First published 1996

A catalogue record for this publication is available from the British Library

ISBN 978-0-521-57605-5 paperback

Transferred to digital printing 2008

CONTENTS

PREFACE

This book contains the results of the research of the authors in the foundations of the theory of definable actions of Polish groups and the associated orbit equivalence relations. It is addressed to graduate students and researchers in this area as well as to mathematicians in other fields, whose work comes in contact with this subject.

The reader will also find some new theorems which are attributed to someone other than the authors. Many people have helped in one way or another to improve this book: by allowing us to include these new theorems; by simplifying some of our proofs; by bringing to our attention some work we were unaware of; or by reading earlier versions of this book – or parts of it – and suggesting improvements in the exposition. We thank all of them: G. Cherlin, W. Comfort, R. Dougherty, G. Hjorth, A. Louveau, M. Megrelishvili, M. G. Nadkarni, R. Sami, S. Solecki, S. Thomas, and an anonymous referee.

The authors would also like to acknowledge the financial support of the National Science Foundation, through grants DMS-9206922 and DMS-9317509, and the help of the Mathematics Department at Caltech in the preparation of this book. We particularly thank J. Cassidy for typing the manuscript.

Howard Becker
Columbia, SC

Alexander S. Kechris
Pasadena, CA

November 1995

INTRODUCTION

A *Polish space* (*group*) is a separable, completely metrizable topological space (group). This book is about actions of Polish groups, in connection with – or from the point of view of – the subject of descriptive set theory. *Descriptive set theory* is the study of definable sets and functions in Polish spaces. The basic classes of definable sets are the classes of Borel, analytic and co-analytic sets, and these certainly constitute the main topic of the book, but at times we also consider other classes of definable sets.

The structure of Borel actions of Polish *locally compact*, i.e., second countable locally compact, topological groups has long been studied in ergodic theory, operator algebras and group representation theory. See, for example, Auslander–Moore [66], Feldman–Hahn–Moore [78], Glimm [61], Kechris [92a], Mackey [57, 62, 89], Moore [82], Ramsay [82, 85], Sinai [89], Varadarajan [63], Vershik–Fedorov [87], Zimmer [84] for a sample of this work. This is closely related to the subject matter of this book. More recently, there has been increasing interest in an extension of the above: studying the structure of Borel actions of *arbitrary*, not necessarily locally compact, Polish groups. Such groups are ubiquitous in mathematics, most often as groups of symmetries of mathematical structures, e.g., the symmetric group S_∞ on a countably infinite set, the group of homeomorphisms $H(X)$ of a compact metrizable space X, the unitary group $U(H)$ of a separable infinite dimensional Hilbert space H, groups of automorphisms of standard measure spaces, diffeomorphism groups, etc. The nonexistence of Haar measure for such groups makes their study quite challenging and requires the development of new methods, one of which is the use of Baire Category techniques.

One area where Borel actions of nonlocally compact groups occur, particularly of the symmetric group S_∞ and its closed subgroups, is in mathematical logic. Of particular importance is the *Vaught Conjecture*, a well-known open problem in logic, and the *Topological Vaught Conjecture* (see Becker [94], Burgess–Miller [75], Lascar [85], Miller [77], Sami [94], Steel [78], Vaught [74]). This falls within the more general framework of *invariant descriptive set theory*, introduced by Vaught [74] and motivated, to a large extent, by mathematical logic. If the term "invariant descriptive set theory" is interpreted in a broad enough fashion, then this is a book on invariant descriptive set theory. However, our perspective is somewhat

different from that of traditional invariant descriptive set theory.

Another place where such actions arise is in the ergodic theory and unitary group representation theory of nonlocally compact groups (see, e.g., Olshanski [82], Vershik [74]). Finally, such actions occur in the theory of operator algebras; see, in particular, Effros [65], [81], two papers motivated by some basic problems in that field which have also played an important role in our work as well.

This book contains a number of results about actions of general Polish groups, some of which are new even in the locally compact case. We will not attempt to describe in detail the contents of the book in this introduction. We do, however, wish to summarize the major themes that we are studying here and the main results related to each of them.

One theme is that of changing the topology on the space being acted upon. One of the main results here is that for a Borel action of a Polish group G on a Polish space X, there is a new Polish topology on X having the same Borel structure such that the action becomes now continuous in the new topology. Equivalently, every Borel action on a Polish space is Borel isomorphic to a continuous one. This answers a question of Ramsay [85], who raised it in the case of locally compact G, and Miller [77]. It was known classically for countable (discrete) groups and was proved recently for $G = \mathbb{R}$ in Wagh [88].

Another result along these lines is that for any continuous action of a Polish group G on a Polish space X and any invariant Borel set $B \subseteq X$, there is a new finer Polish topology on X (which therefore has the same Borel structure) in which B is now clopen and the action remains continuous. This result was known for countable groups and for S_∞.

A second theme is that of *universal actions.* We establish the existence of universal actions for any given Polish group, and, moreover, we show that the universal action can be taken to be a continuous action on a compact Polish space. This extends a result of Mackey [62] and Varadarajan [63] for the locally compact case.

These first two themes appear in §5 and §2, respectively. There are several applications of these results to various topics. A classical theorem of Tarski asserts that for any action of a group G on a set X, there is a G-invariant finitely additive probability measure on X iff there is no G-paradoxical decomposition of X. The problem has been raised (see, e.g., Wagon [93] for the history of this problem) whether there is an analog of Tarski's Theorem for countably additive probability measures. We establish in §4, using also some recent work of Nadkarni [90], the following analog: for a Borel action of a Polish group on a Polish space, there is a G-invariant

probability Borel measure iff there is no countable G-paradoxical Borel decomposition of X. In view of known counterexamples, this appears to be an optimal analog of Tarski's Theorem.

In another direction, we show in §6 that various forms of the Topological Vaught Conjecture, originally proposed by Miller (see Rogers *et al.* [80], p. 484) turn out to be equivalent. One form is that in a continuous action of a Polish group on a Polish space there are either countably many or perfectly many orbits. Other forms assert the same conclusion for Borel actions or continuous actions restricted to Borel invariant sets. We also show in §6 that the Topological Vaught Conjecture for the symmetric group S_∞ is equivalent to the usual model theoretic Vaught Conjecture for the language $L_{\omega_1\omega}$ (which asserts that any sentence in this language has either countably many or perfectly many countable models, up to isomorphism.) Both the Vaught Conjecture (even for first-order theories) and the Topological Vaught Conjecture are of course still open.

A third theme is that of the *logic actions*. These are particular continuous actions by the group S_∞ on *codes* for countable L-structures, where the orbit equivalence relation is isomorphism. The logic actions serve three purposes: as a means of applying the study of group actions to questions in model theory; as motivation for generalizing concepts and theorems from model theory (that is, from logic actions) to other actions; and, as we show in §2, as concrete examples of the universal S_∞-action.

The fourth theme is that of *dichotomies* for the orbit space, that is, theorems to the effect that the orbit space is either "small" or else contains a copy of some specific "large" set. The Vaught Conjecture asserts such a dichotomy, and, indeed, that conjecture is what motivated quite a bit of the research in this book. Other important dichotomies, discussed in §3, are the Silver dichotomy and the Glimm–Effros type dichotomies.

The fifth theme is that of *descriptive complexity* for the orbit equivalence relation induced by a Borel action of a Polish group on a Polish space. In general such an equivalence relation is analytic but not Borel. We show though in §7 that the space acted upon can be canonically decomposed into an $\aleph_1$ sequence of invariant Borel sets on which the orbit equivalence is Borel and this sequence is cofinal among such invariant Borel sets. Finally, we characterize the Borelness of the orbit equivalence relation in terms of the complexity of the stabilizer function (from points in the space to closed subgroups of the Polish group): It is Borel iff the stabilizer function is Borel.

The sixth and final theme is that of the "*definable cardinality*" of the orbit space of an action. This is related to some of the earlier themes, as for example that of dichotomies, and is developed in more detail in §8.

Some of the results contained here first appeared, without complete proofs, in the research announcement Becker–Kechris [93]. (In our research announcement we stated that we did not know whether the universal action can always be taken to be a continuous action on a *compact* space; as pointed out above, this has now been solved.) Some of them have also been previously announced in Becker [94] and Kechris [94].

This book contains a fair amount of background material – definitions and known theorems, sometimes with proofs, sometimes without. These preliminaries are introduced when needed, and so are scattered throughout the book. In particular, most of §1 and of the first halves of §2, 3 survey the basic facts about Polish groups, actions of Polish groups, and the induced equivalence relations, respectively, that are needed in the sequel.