This book provides the first self-contained comprehensive exposition of the theory of dynamical systems as a core mathematical discipline closely intertwined with most of the main areas of mathematics. The authors introduce and rigorously develop the theory while providing researchers interested in applications with fundamental tools and paradigms.

The book begins with a discussion of several elementary but fundamental examples. These are used to formulate a program for the general study of asymptotic properties and to introduce the principal theoretical concepts and methods. The main theme of the second part of the book is the interplay between local analysis near individual orbits and the global complexity of the orbit structure. The third and fourth parts develop in depth the theories of low-dimensional dynamical systems and hyperbolic dynamical systems.

The book is aimed at students and researchers in mathematics at all levels from advanced undergraduate up. Scientists and engineers working in applied dynamics, nonlinear science, and chaos will also find many fresh insights in this concrete and clear presentation. It contains more than four hundred systematic exercises.

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

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Introduction to the Modern Theory of Dynamical Systems

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Anatole Katok and Boris Hasselblatt
Frontmatter
More information

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

18 H. O. Fattorini *The Cauchy problem*
19 G. G. Lorentz, K. Jetter, and S. D. Riemenschneider *Birkhoff interpolation*
21 W. T. Tutte *Graph theory*
22 J. R. Bastida *Field extensions and Galois theory*
23 J. R. Cannon *The one-dimensional heat equation*
25 A. Salomaa *Computation and automata*
26 N. White (ed.) *Theory of matroids*
27 N. H. Bingham, C. M. Goldie, and J. L. Teugels *Regular variation*
28 P. P. Petrushev and V. A. Popov *Rational approximation of real functions*
29 N. White (ed.) *Combinatorial geometries*
30 M. Pohst and H. Zassenhaus *Algorithmic algebraic number theory*
31 J. Aczel and J. Dhombres *Functional equations containing several variables*
32 M. Kuczma, B. Chozewski, and R. Ger *Iterative functional equations*
33 R. V. Ambartzumian *Factorization calculus and geometric probability*
34 G. Gripenberg, S.-O. Londen, and O. Staffans *Volterra integral and functional equations*
35 G. Gasper and M. Rahman *Basic hypergeometric series*
36 E. Torgersen *Comparison of statistical experiments*
37 A. Neumaier *Interval methods for systems of equations*
38 N. Korneichuk *Exact constants in approximation theory*
39 R. A. Brualdi and H. J. Ryser *Combinatorial matrix theory*
40 N. White (ed.) *Matroid applications*
41 S. Sakai *Operator algebras in dynamical systems*
42 W. Hodges *Model theory*
43 H. Stahl and V. Totik *General orthogonal polynomials*
44 R. Schneider *Convex bodies*
45 G. Da Prato and J. Zabczyk *Stochastic equations in infinite dimensions*
46 A. Bjorner, M. Las Vergnas, B. Sturmfels, N. White, and G. Ziegler *Oriented matroids*
47 E. A. Edgar and L. Sucheston *Stopping times and directed processes*
48 C. Sims *Computation with finitely presented groups*
49 T. Palmer *Banach algebras and the general theory of *-algebras*
50 F. Borceux *Handbook of categorical algebra I*
51 F. Borceux *Handbook of categorical algebra II*
52 F. Borceux *Handbook of categorical algebra III*

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Introduction to the Modern Theory of Dynamical Systems

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Frontmatter
More information

Contents

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Frontmatter
More information

Preface

The theory of dynamical systems is a major mathematical discipline closely intertwined with most of the main areas of mathematics. Its mathematical core is the study of the global orbit structure of maps and flows with emphasis on properties invariant under coordinate changes. Its concepts, methods, and paradigms greatly stimulate research in many sciences and have given rise to the vast new area of applied dynamics (also called nonlinear science or chaos theory). The field of dynamical systems comprises several major disciplines, but we are interested mainly in finite-dimensional differentiable dynamics. This theory is inseparably connected with several other areas, primarily ergodic theory, symbolic dynamics, and topological dynamics. So far there has been no account that treats differentiable dynamics from a sufficiently comprehensive point of view encompassing the relations with these areas. This book attempts to fill this gap. It provides a self-contained coherent comprehensive exposition of the fundamentals of the theory of smooth dynamical systems together with the related areas of other fields of dynamics as a core mathematical discipline while providing researchers interested in applications with fundamental tools and paradigms. It introduces and rigorously develops the central concepts and methods in dynamical systems and their applications to a wide variety of topics.

What this book contains. We begin with a detailed discussion of a series of elementary but fundamental examples. These are used to formulate the general program of the study of asymptotic properties as well as to introduce the principal notions (differentiable and topological equivalence, moduli, structural stability, asymptotic orbit growth, entropies, ergodicity, etc.) and, in a simplified way, a number of important methods (fixed-point methods, coding, KAM-type Newton method, local normal forms, homotopy trick, etc.).

The main theme of the second part is the interplay between local analysis near individual (e.g., periodic) orbits and the global complexity of the orbit structure. This is achieved by exploring hyperbolicity, transversality, global topological invariants, and variational methods. The methods include the study of stable and unstable manifolds, bifurcations, index and degree, and construction of orbits as minima and minimaxes of action functionals.

In the third and fourth parts the general program outlined in the first part is carried out to considerable depth for low-dimensional and hyperbolic dynamical systems which are particularly amenable to such analysis. Hyperbolic systems are the prime example of well-understood complexity. This manifests itself in an orbit structure that is rich both from the topological and statistical point of view and stable under perturbation. At the same time the principal features can be described qualitatively and quantitatively with great precision. In low-dimensional dynamical systems on the other hand there are two situations. In the "very low-dimensional" case the orbit structure is simplified and admits only a limited amount of complexity. In the "low-dimensional" case some complexity is possible, yet additional major aspects of the orbit structure can be understood via hyperbolicity or related types of behavior.

Although we develop most themes related to differentiable dynamics in some depth we have not tried to write an encyclopedia of differentiable dynamics. Even if this were possible, the resulting work would be strictly a reference source and not useful as an introduction or a text. Consequently we also do not strive to present the most definitive results available but rather to provide organizing principles for methods and results. This is also not a book on applied dynamics and the examples are not chosen from those models that are widely studied in various disciplines. Instead our examples arise naturally from the internal structure of the subject and contribute to its understanding. The emphasis placed on various areas in the field is not dictated by the relative amount of published work or research activity in those areas, but reflects our understanding of what is basic and fundamental in the subject. An obvious disparity appears in the area of one-dimensional (real and especially complex) dynamics, which witnessed a great surge of activity in the past 15 years producing a number of brilliant results. It plays a relatively modest role in this book. Real one-dimensional dynamics is used mainly as an easy model situation in which various methods can be applied with considerable success. Complex dynamics, which is in our view a fascinating but rather specialized area, appears only as a source of examples of hyperbolic sets. On the other hand we try to point out and emphasize the interactions of dynamics with other areas of mathematics (probability theory, algebraic and differential topology, geometry, calculus of variations, etc.) even in some situations where the current state of knowledge is somewhat tentative.

How to use this book. This book can be used both as a text for a course or for self-study and as a reference book. As a text it would most naturally be used as the primary source for graduate students with background equivalent to one year of graduate study at a major U.S. university who are interested in becoming specialists in dynamical systems or want to acquire solid general knowledge of the field. Some portions of this book do not assume as much background and can be used by advanced undergraduate students or graduate students in science and engineering who want to learn about the subject without becoming experts. Those portions include Chapter 1, most of Chapters 2, 3, and 5, parts of Chapters 4, 6, 8, and 9, Chapters 10 and 11, and most of Chapters 12, 14, 15, and 16. The 472 exercises are a very important part of the book. They fall into several categories. Some of them directly illustrate the use of results or methods from the text; others explore examples that are not discussed in the text or indicate further developments. Sometimes an important side topic is developed in a series of exercises. Those 317 that we do not consider routine have been provided with hints or brief solutions in the back of the book. An asterisk indicates our subjective assessment of higher difficulty, due to the need for either inventiveness or familiarity with material not obviously related to the subject at hand.

Each of the four parts of the book can be the basis of a course roughly at the second-year graduate level running one semester or longer. From this book one can tailor many courses dedicated to more specialized topics, such as variational methods in classical mechanics, hyperbolic dynamical systems, twist maps and applications, an introduction to ergodic theory and smooth ergodic theory, and the mathematical theory of entropy. In order to assist both students and teachers in selecting material for a course we summarize the principal interrelations between the chapters in Figure F.1. A solid arrow A⟶B indicates that a major portion of the material from Chapter A is used in Chapter B (this relation is transitive). A dashed arrow A-->B indicates that material from Chapter A is used in some parts of Chapter B.

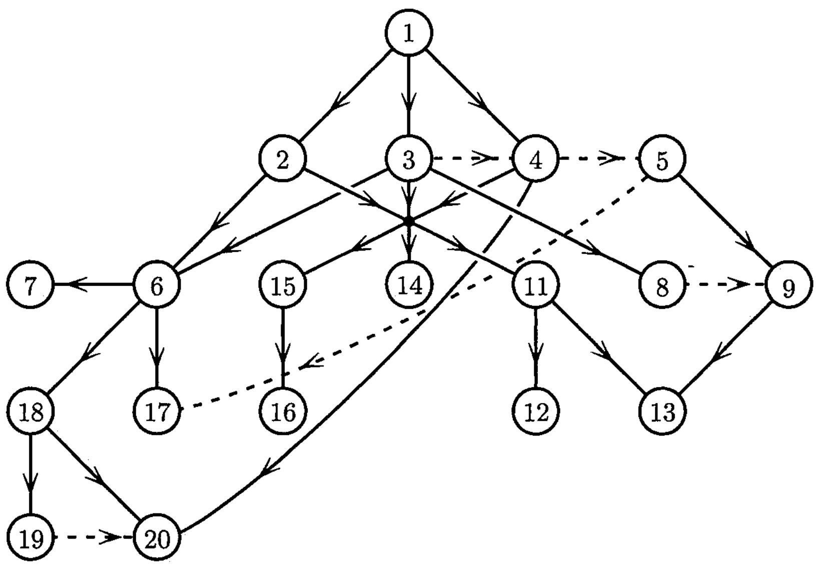

FIGURE F.1.

With the exception of Chapters 1–4, which form a common basis for the rest of the book, generally the material in the left part of the diagram deals with hyperbolic dynamics, that in the middle with low-dimensional dynamics, and that in the right with aspects of differentiable dynamics related to topology and classical mechanics.

There are various kinds of material used in this book. First of all we tacitly use, and assume familiarity with, the results of linear algebra (including Jordan normal forms), calculus of several variables, the basic theory of ordinary differential equations (including systems), elementary complex analysis, basic set theory, elementary Lebesgue integration, basic group theory, and some Fourier series. There is a next higher level of essential background material which is reviewed in the appendix. Most of the material in the appendix is of this nature, namely, the standard theory of topological, metric, and Banach spaces, elementary homotopy theory, the basic theory of differentiable manifolds including vector fields, bundles and differential forms, and the definition and basic properties of Riemannian metrics. Some topics are used on isolated occasions only. This last level of material includes the basic topology and geometry of surfaces and the general theory of measures, σ-algebras, and Lebesgue spaces, homology theory, material related to Lie groups and symmetric spaces, curvature and connections on manifolds, transversality, and normal families of complex functions. Most, but not all, of this material is also reviewed in the appendix, usually in a less detailed fashion. Either such material can be taken on faith without loss to the application in the text, or otherwise the pertinent portion of the text can be skipped without great loss.

On several occasions we include an important background fact without proof in the text. This happens when a certain result is organically related to a particular section. The Lefschetz Fixed-Point Formula is a good example of such a result.

Sources. Most of the material in this book does not consist of original results. Nevertheless the presentation of most of the material is our own and consists of original or considerably modified proofs of known results, explanations of the structure and interconnectedness of the subject, and so forth. Some portions of the text, roughly a sixth of it, mostly in Parts 3 and 4, closely follow other published sources, the majority of these being original research articles. An outstanding example is the presentation of portions of the hyperbolic theory in Chapters 18 and 20 which was given such a clear treatment by R. Bowen in his articles in the seventies that it could hardly be improved. On several occasions we follow the exposition of a subject in existing books. With the exception of some basic subjects such as Hamiltonian formalism or variational calculus this occurs only in Part 3. The reason for this is that low-dimensional dynamics has a much better developed expository literature than the field as a whole. We acknowledge all borrowings of proofs and presentations of which we are aware in the notes near the end of the book.

Since we aim to present the subject by developing it from first principles and in a self-contained way, rather than to give an exhaustive account of the development and current state of the field, we do not attempt a comprehensive listing of relevant references which would easily increase our bibliography to a thousand items or more. In particular, not all theorems are attributed to the original authors, especially if the results are part of the broader developments in the field, rather than landmark results or of a rather special nature. Most of the attributions are relegated to the notes. These consist of general comments arranged by section and some numbered remarks to particular points in the main text. Furthermore, in order not to interrupt the logical flow of the text all bibliographical references for the main part of the book are also relegated to the notes. Our historical comments, in both the introduction and the notes, do not aim to present a coherent account of the development of the subject, but to subjectively select some of its major moments.

We have included several types of literature in the bibliography. First, we have tried to list all major monographs and representative textbooks and surveys covering the principal branches of dynamics. Next there are landmark papers that introduce and develop various branches of our subject, define principal notions, or contain proofs of major results. We try to list all the sources on which the presentation in various parts of the book is based, or that inspired our presentation in other places and many (but not all) of the original sources for specific results presented in the text. Finally there is a sample of references to important work, both original and surveys, in some areas touched upon but not treated in the text. According to our principle of selecting models for their intrinsic interest rather than their value for concrete scientific problems we omit works by nonmathematicians (even important ones) that are dedicated to the study of models motivated by scientific problems so long as these contain only hypotheses and numerical results. References to such works are widely available in many of the books and surveys that we quote.

History and acknowledgments. The general idea of writing a broad introduction to the theory of dynamical systems first occurred to the first author when he taught a graduate course at the California Institute of Technology in 1984–5. This course resulted in two sets of lecture notes prepared by the second author and by his fellow graduate student John Lindner to whom we are deeply grateful. The key idea of introducing the principal notions and methods via a presentation of a series of basic examples crystallized when the first author was preparing and teaching an intensive four-week course in July 1986 at the Summer Mathematics Institute for graduate students at Fudan University in Shanghai. The summary and notes from that course became the germ for major parts of Chapters 1–4. Further progress was made during another graduate course at the California Institute of Technology in 1986–7 after which it became clear that the original project of a book of 300–350 pages would result in too sketchy and incomplete an account of the subject. In the summer of 1989 we developed a detailed plan of the book which has been carried out with some substantial later modification. Another graduate course during the first author's first year at the Pennsylvania State University (1990–1) helped to test some existing parts of the book and develop some new material.

We feel deep gratitude to the California Institute of Technology, Tufts University, and the Pennsylvania State University for providing excellent working conditions and supporting several mutual visits. Special thanks are due to the Mathematical Sciences Research Institute in Berkeley, California, where we worked together on major portions of the book in the summer of 1992. During this period our project was transformed from a collection of drafts into an incomplete but coherent product.

We also owe thanks and gratitude to numerous individuals for providing various kinds of help and inspiration during this project. We apologize for any omissions of people whose comments and suggestions may have been incorporated and forgotten.

Jessica Madow, the technical typist at the California Institute of Technology, typed major portions of the then existing manuscript in Exp. Kathy Wyland and Pat Snare at the Pennsylvania State University typed the first drafts of

many chapters in TEX. Several people helped with computer support or typesetting advice. David Glaubman at the Mathematical Sciences Research Institute was very helpful, Michael Downes at the Technical Support Department of the American Mathematical Society helped to make the running heads come out right on every single page, and our colleague Uwe Schmock at the ETH wrote the overbar macro for TEX and made other useful comments. Boris Katok made the majority of the illustrations for the book. Bill Schlesinger gave us the initial tutoring that enabled us to make numerous pictures using Matlab. We are deeply grateful to the editors of Cambridge University Press: David Tranah who encouraged and prodded us during the earlier stages of the project and Lauren Cowles who patiently guided us through the process of finishing the book and getting it ready for production. This book was typeset in TEX using AMS-TEX, the TEX macro package of the American Mathematical Society.

Viorel Niţică and Alexej Kononenko wrote solutions to the majority of the exercises. Their work helped to correct some flawed exercises, and we used their solutions to write many of our hints.

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A number of corrections were made between printings. We would like to thank Luis Barreira, Marlies Gerber, Karl Friedrich Siburg, Garrett Stuck and Andrew Török who pointed out many small errors. Peter Walters found inaccuracies in Lemma 4.5.2 and Lemma 20.2.3. Robert McKay pointed out that some results of Section 14.2 needed recurrence hypotheses and Jonathan Robbins noted problems with the first version of Step 5 in the proof of the Hadamard-Perron Theorem 6.2.8. Tim Hunt corrected the DA construction. Corrections are listed at `http://www.tufts.edu/~bhasselb/thebook.html`.

A serious omission survived three printings: Section 20.6 is entirely due to Charles Toll, an attribution we inadvertently failed to make. Our sincere apologies.

Last, and most importantly, we wish to thank Svetlana Katok and Kathleen Hasselblatt for constant support and inspiration.