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Introduction

1. Principal branches of dynamics

The most general and somewhat vague notion of a dynamical system includes the following ingredients:

(i). A "phase space" X, whose elements or "points" represent possible states of the system.

(ii). "Time", which may be discrete or continuous. It may extend either only into the future (irreversible or noninvertible processes) or into the past as well as the future (reversible or invertible processes). The sequence of time moments for a reversible discrete-time process is in a natural correspondence to the set of all integers; irreversibility corresponds to considering only nonnegative integers. Similarly, for a continuous-time process, time is represented by the set of all real numbers in the reversible case and by the set of nonnegative real numbers for the irreversible case.

(iii). The time-evolution law. In the most general setting this is a rule that allows us to determine the state of the system at each moment of time t from its states at all previous times. Thus, the most general time-evolution law is time dependent and has infinite memory. In the course of this book, however, we will consider only those evolution laws that allow us to define all future (and for reversible systems also past) states given a state at any particular moment. Furthermore we will assume that the law of time evolution itself does not change with time. In other words, the result of time evolution will depend only on the initial position of the system and on the length of the evolution but not on the moment when the state of the system was initially registered. Thus, if our system was initially at a state $x \in X$, it will find itself after time t at a new state, which is uniquely determined by x and t, and thus can be denoted by $F(x,t)$. Fixing t, we obtain a transformation $\varphi^t \colon x \mapsto F(x,t)$ of the phase space into itself. These transformations for different t are related to each other. Namely, the evolution of the state x for time $s+t$ can be accomplished by first applying the transformation φ^t to x and then by applying φ^s to the new state $\varphi^t(x)$. Thus, we have $F(x, t+s) = F(\varphi^t(x), s)$ or equivalently, the transformation φ^{t+s} is equal to the composition of φ^t and φ^s. In other words, the transformations φ^t form a semigroup. For a reversible system the transformations φ^t are defined for both positive and negative values of t and each φ^t is invertible. Thus, a reversible discrete-time dynamical system is represented by a cyclic group $\{F^n = (\varphi^1)^n \mid n \in \mathbb{Z}\}$ of one-to-one transformations of the phase space onto itself. Similarly, a reversible continuous-time dynamical system determines a one-parameter group $\{\varphi^t \mid t \in \mathbb{R}\}$ of one-to-one transformations of X onto itself.

The most characteristic feature of dynamical theories, which distinguishes them from other areas of mathematics dealing with groups of automorphisms of various mathematical structures, is the emphasis on asymptotic behavior, especially in the presence of nontrivial recurrence, that is, properties related with the behavior as time goes to infinity. The best way to explain what significant asymptotic properties are is to examine specific examples of dynamical systems and to determine the most characteristic features of their behavior. We will do that in Chapter 1 and then we will summarize some of our findings and present a list of interesting properties in Sections 3.1, 3.3, 4.1, 4.2d, and 4.3. This summary is preceded by an examination of natural equivalence relations for dynamical systems in Chapter 2 which sets the stage for treating asymptotic properties as invariants of those equivalence relations.

Historically, smooth continuous-time dynamical systems appeared first because of Newton's discovery that the motions of mechanical objects can be described by second-order ordinary differential equations. More generally, many other natural and social phenomena, such as radioactive decay, chemical reactions, population growth, or dynamics of prices on the market, may be modeled with various degrees of accuracy by systems of ordinary differential equations. These situations fit into the domain of our investigation if there is no explicit time-dependence in the coefficients and right-hand parts of the equations.

In virtually all situations of interest the phase space of a dynamical system possesses a certain structure which the evolution law respects. Different structures give rise to theories dealing with dynamical systems that preserve those structures. Let us mention the most important of those theories.

1. Ergodic theory. Here the phase space X is a "good" measure space, that is, a Lebesgue space (cf. Section 6 of the Appendix) with a finite or σ-finite measure μ. We can consider as a structure in X either the measure μ itself or its equivalence class which is determined by the collection of all sets of measure zero. Accordingly, ergodic theory concerns groups or semigroups of measurable transformations of X that either preserve μ or transform it into an equivalent measure. In the latter case the measure μ is called *quasi-invariant*. In this book ergodic theory plays an important but auxiliary role. It provides the appropriate paradigms and tools for studying asymptotic distribution and statistical behavior of orbits for smooth dynamical systems. Some central concepts and results of ergodic theory are introduced and discussed in Chapter 4.

The origins of ergodic theory go back to the famous ergodic hypothesis of Boltzmann who postulated equality of time averages and space averages for systems in statistical mechanics. Within mathematics the notions of ergodic theory arose from the study of uniform distributions of sequences. The Kronecker–Weyl Equidistribution Theorem (Proposition 4.2.1) is an early example of such a result. H. Poincaré observed that the preservation of a finite invariant measure forces strong conclusions about recurrence which are encapsulated in his Recurrence Theorem (Theorem 4.1.19). The systematic development of ergodic theory as a mathematical subject started around 1930 by von Neumann who looked at the subject primarily from a functional-analytic viewpoint. Among the early major contributors to the subject were G. D. Birkhoff, E. Hopf, and S. Kakutani. The critical point in the development of ergodic theory which forever changed the emphasis from the functional-analytic to the probabilistic and later geometric and combinatorial viewpoints was the introduction of entropy by A. Kolmogorov around 1958. It built upon C. Shannon's seminal development of information theory which was given the appropriate mathematical treatment by A. Khinchin. Kolmogorov's work was quickly followed by the development of an entropy theory based on the probabilistic viewpoint primarily by Y. Sinai and V. Rokhlin which culminated in Sinai's weak isomorphism theorem. The next crucial juncture was the first proof of the isomorphism of Bernoulli shifts of equal entropy which was obtained by D. Ornstein via combinatorial constructions. This work was followed by the development of the isomorphism theory which in particular gave necessary and sufficient conditions for metric isomorphism to a Bernoulli shift. Among later major developments one should note the Kakutani (monotone) equivalence theory, H. Furstenberg's theory of multiple recurrence, and the finitary isomorphism theory.

2. Topological dynamics. The phase space in this theory is a good topological space, usually a metrizable compact or locally compact space (see Section 1 of the Appendix). Topological dynamics concerns itself with groups of homeomorphisms and semigroups of continuous transformations of such spaces. Sometimes these objects are called topological dynamical systems. Similarly to the case of ergodic theory we use in this book notions and results from topological dynamics primarily as a framework and a tool for studying smooth dynamical systems. Though we are not making an attempt to provide a comprehensive introduction to the field, a fair amount of material from topological dynamics appears in this book, beginning with our first survey of examples in Chapter 1 and then in Chapter 3. Sections 4.1, 4.5 and later 20.1 and 20.2 provide crucial links between topological dynamics and ergodic theory. Some material in Chapter 8 (for example, Theorem 8.3.1) as well as all of Chapters 11 and 15 deal with particular classes of dynamical systems without any differentiability assumptions and thus belong to topological dynamics.

Topological dynamics was founded by Poincaré when he introduced the idea of qualitatively describing the solutions of differential equations that could not be solved analytically. One of his early achievements was the classification of circle maps (Theorem 11.2.7). M. Morse and G. D. Birkhoff made major contributions to topological dynamics in the process of trying to understand more classical systems (behavior of geodesics and Hamiltonian systems). Later a more intrinsic approach was developed by G. Hedlund, J. Oxtoby, and others. An important subject in topological dynamics is H. Furstenberg's theory of distal extensions which was further developed by R. Ellis.

3. The theory of smooth dynamical systems or differentiable dynamics. As the name suggests, the phase space here possesses the structure of a smooth manifold, for example, a domain or a closed surface in a Euclidean space (see Section 3 of the Appendix for a more detailed description). This theory, which is the prime subject of this book, concerns diffeomorphisms and flows (smooth one-parameter groups of diffeomorphisms) of such manifolds and iterates of noninvertible differentiable maps. In this book we will deal mostly with finite-dimensional situations. Interest in infinite-dimensional dynamical systems has been growing steadily during the past two decades, to a large extent stimulated by problems in fluid dynamics, statistical mechanics, and other fields of mathematical physics. Several directions in infinite-dimensional dynamics have been developed to a considerable extent starting from analogies with various branches in finite-dimensional dynamics.

Since a finite-dimensional smooth manifold possesses a natural locally compact topology, the theory of smooth dynamical systems naturally draws upon notions and results from topological dynamics. Another deeper reason for these interrelations arises from the fact that in dealing with asymptotic behavior of smooth dynamical systems one is likely to encounter very complicated nonsmooth phenomena, which in other contexts would be dismissed as pathological. In particular, some important invariant sets for a smooth system, for example, attractors (Definition 3.3.1), may not have any smooth structure and consequently, such sets should be studied from a different, nonsmooth, point of view. *Symbolic dynamics*, the study of a specific class of topological dynamical systems which occur as closed invariant subsets of the shift transformation in a sequence space (cf. Section 1.9), is particularly important in that respect. For further motivation of the relationships between topological and smooth dynamics see Section 2.3.

Relations with ergodic theory are also intimate, both because invariant measures provide a powerful tool for the study of asymptotic properties of smooth dynamical systems and because the smooth structure on a finite-dimensional manifold determines a natural class of quasi-invariant measures for differentiable dynamical systems (see Section 5.1).

Sometimes the part of the theory of smooth dynamical systems that concerns measure-theoretic properties of such systems is given the separate name *smooth ergodic theory*. One might also say that smooth ergodic theory is the study of automorphisms of a composite structure formed by a smooth manifold and a reasonable measure on it. Chapter 20 and the Supplement are dedicated to this subject. A number of results belonging to smooth ergodic theory are scattered among the earlier chapters.

Poincaré is also the father of differentiable dynamics. His main contribution was to emphasize the qualitative approach as opposed to the traditional emphasis on explicit solutions of differential equations of mechanics. His other achievement was the founding of the local theory of maps and vector fields near fixed and periodic orbits (cf. Sections 2.1, 6.3, 6.6). Other principal figures in the early stages of the field were A. M. Lyapunov and J. Hadamard who introduced various concepts of stability and developed major analytic tools (for example, the Hadamard–Perron Theorem 6.2.8). Part of Poincaré's program was carried out by G. D. Birkhoff who proved, among other things, Poincaré's celebrated "Last Geometric Theorem" which gives a mechanism responsible for dynamical complexity in mechanical systems with two degrees of freedom. Another aspect of Poincaré's program was developed by A. Denjoy who introduced some key new ideas in the process of completing Poincaré's theory of circle maps and flows on the two-dimensional torus. Symbolic dynamics appeared as a very useful tool beginning with a seminal paper by E. Artin and it was greatly developed by Morse and Hedlund. E. Hopf was the first to realize that hyperbolicity is a key mechanism that produces complicated behavior in nonlinear dynamical systems. His proof of ergodicity of the geodesic flow of surfaces of negative curvature can be viewed as the first major result in smooth ergodic theory.

Another principal root of the modern global approach to the study of smooth dynamical systems was the notion of structural stability which was first introduced by A. Andronov and L. S. Pontryagin in the study of flows on surfaces and later developed in that setting by Peixoto. It was given a second life by Smale who discovered that systems with complicated orbit behavior (the "horseshoe", Section 2.5) can be structurally stable. Subsequently Smale, Anosov, Sinai, and Bowen developed the core of the theory of hyperbolic dynamical systems. They greatly developed methods from ergodic theory and topological dynamics due to Hopf and Hedlund as well as more classical ideas going back to Hadamard, Perron, and Lyapunov. Identifying a certain hyperbolicity as sufficient (J. Robbin, C. Robinson) and necessary (R. Mañé) for structural stability was one of the crowning achievements of the theory of smooth dynamical systems. A major impetus to smooth ergodic theory was given by D. Ruelle and Y. Sinai who introduced ideas and methods from statistical mechanics to the theory of smooth dynamical systems. The next important step was made by Y. B. Pesin who developed the general structural theory of smooth measure-preserving systems based on the concept of nonuniform hyperbolicity. We should also mention the work of M. Herman and J.-C. Yoccoz on smooth classification of circle diffeomorphisms and the work of D. V. Anosov and A. Katok on constructions of smooth dynamical systems with various often unexpected properties.

4. Hamiltonian or symplectic dynamics. This theory is a natural generalization of a study of differential equations of classical mechanics. The phase space here is an even-dimensional smooth manifold with a nondegenerate closed differential 2-form Ω. One-parameter groups of Ω-preserving diffeomorphisms correspond to Hamiltonian differential equations in classical mechanics. An individual Ω-preserving diffeomorphism generalizes the notion of a canonical transformation. We first encounter such systems in Section 1.5 and return to this field in a more systematic way in Section 5.5.

The origin of Hamiltonian dynamics as an object of study from the point of view of dynamical systems is largely in the questions of celestial mechanics. Again Poincaré introduced the fundamental approach of the qualitative study of the n-body problem. Later two distinct directions of study emerged: (i) the investigation of dynamical complexity in this problem due to some hyperbolicity (Alekseev, Conley) and (ii) the study of integrable systems and their perturbations which led to the KAM theory. Though both the hyperbolic and integrable paradigm were available since Poincaré, it was Kolmogorov's profound contribution to realize that many qualitative features of (the very exceptional) integrable systems persist to some extent under perturbations and appear also in generic situations (for example, near an elliptic fixed point). Both of these lines of thought were influenced by the question of the stability of the solar system which was addressed by the hyperbolic approach in terms of the stability of an n-body system and by the KAM approach by considering perturbations, for example, of the (integrable) central force problem without interactions between planets. The work of Conley and Zehnder established a synthesis of topological and variational methods which became the cornerstone of modern global symplectic geometry. A renaissance of the study of completely integrable systems started with a seminal paper by Gardner, Greene, Kruskal, and Miura and the discovery by P. Lax of new mechanisms for producing integrable systems. It led both to a proliferation of new interesting examples of finite-dimensional integrable systems as well as to the theory of infinite-dimensional Hamiltonian systems whose applications to nonlinear partial differential equations were a major breakthrough by providing for the first time means for a complete qualitative analysis in situations other than those with the most simple asymptotic behavior.

2. Flows, vector fields, differential equations

The description of a dynamical system is somewhat easier when time is discrete, because the map generating a discrete-time system often can be given explicitly, usually by means of some formulas. In contrast, a continuous-time dynamical system is usually given infinitesimally (for example, by means of differential equations) and the reconstruction of the dynamics from this infinitesimal description involves some kind of integration process. In this and the next section we will very briefly discuss this local (in time) aspect of the theory of continuous-time dynamical systems and some simple relations between the discrete-time and the continuous-time situations.

We assume that the phase space is a smooth manifold of dimension m which we will usually denote by M, and thus our time evolution is given by a smooth function $F(x,t) = \varphi^t(x)$, $x \in M$, $t \in \mathbb{R}$, which satisfies the group (composition) property $\varphi^t \circ \varphi^s = \varphi^{t+s}$ and may or may not be defined for all x and t. Let us consider first the local aspect of the situation. When we fix $x \in M$ and vary t we obtain a parameterized smooth curve on M. Let $\xi(x)$ be the tangent vector to this curve at $t = 0$, that is, at the point x. Properly speaking, the vector $\xi(x)$ belongs to the tangent space T_xM which is an m-dimensional linear space "attached" to M at the point x. The map $x \mapsto \xi(x)$ forms a section of the tangent bundle $TM = \bigcup_{x\in M} T_xM$ or a *vector field* on M (see Section 3 of the Appendix for more details). Of course, the local version of this construction is familiar to everybody who completed a standard course of advanced calculus. Namely, let $U \subset M$ be a coordinate neighborhood with coordinates $(s_1, \dots, s_m)$. Then the tangent bundle TU is simply a direct product $U \times \mathbb{R}^m$ and a vector field is determined by a map from U to $\mathbb{R}^m$, that is, by m real-valued functions $v_1, \dots, v_m$, as follows. Denoting by $\dfrac{\partial}{\partial s_i}$ the basic vector fields which associate to every point the ith vector of the standard basis in $\mathbb{R}^m$ we can represent every vector field locally as $\sum_{i=1}^m v_i(s_1, \dots, s_n)\dfrac{\partial}{\partial s_i}$. If our initial point x is represented by coordinates $s_1^0, \dots, s_m^0$ then the evolution of this point is obtained by solving the system of first-order ordinary differential equations

$$\frac{ds_i}{dt} = v_i(s_1, \dots, s_m)$$

with initial conditions $s_i(0) = s_i^0$ for $i = 1, \dots, m$.

We know from the standard theory of ordinary differential equations that under very moderate smoothness assumptions, for example, if the functions v_i are continuously differentiable, the solution for sufficiently small time exists, is unique, and depends smoothly on the initial condition.

Thus, at least for small values of t, the transformation φ^t can be recovered from the vector field. For larger t one should take compositions of maps defined in local coordinates. If solutions exist for all real values of t, the vector field is called *complete*. We should keep in mind that on a manifold we have to work in different local coordinate systems if t is large, but this does not present any difficulties. If the manifold M is compact and has no boundary then it can be covered by a finite number of coordinate charts. Inside any chart the solutions exist for a fixed length of time. Since every point $x \in M$ belongs to a coordinate neighborhood which is not very small, this implies that any C^1 vector field on a closed compact manifold without boundary is complete and thus defines a *smooth flow*, that is, a one-parameter group of diffeomorphisms of M.

This is one of the reasons why we will often prefer to consider dynamical systems on compact manifolds. This preference will not be universal because in many situations such as local and semilocal problems (cf. Section 0.4 and Chapter 6) or systems of differential equations associated to many concrete mechanical and other problems, this assumption would be too restrictive.

Exercise

0.2.1. *Show (in detail) that a smooth vector field on a compact manifold is complete.*

3. Time-one map, section, suspension

There are several useful relations between continuous-time and discrete-time dynamical systems.

The most obvious way to associate a discrete-time system to a flow $\{\varphi^t\}_{t\in\mathbb{R}}$ is to take the iterates of the map φ^{t_0} for some value of t_0, say, $t_0 = 1$. However, only very few diffeomorphisms may be obtained that way. For example, let $f = \varphi^{t_0}$ and assume that $f^k(x) = x$, where $k > 1$, but $f(x) \neq x$ so that the orbit of x is periodic, but not fixed. But then for every $t \in \mathbb{R}$

$$f^k\varphi^t(x) = \varphi^{kt_0+t}(x) = \varphi^t(\varphi^{kt_0}(x)) = \varphi^t(f^k(x)) = \varphi^t(x).$$

Hence every point $\varphi^t(x)$ is also a periodic point of period k for f. Thus if f has an isolated periodic point of period greater than one, the map f cannot be obtained as the time-t map of any flow.

Another more local but also more useful method is the construction of a *Poincaré (first-return) map*. Let us take a point $x \in M$ such that $\xi(x) \neq 0$ and an $(m-1)$-dimensional (codimension-one) submanifold N containing x and transversal to the vector field. The latter property simply means that for every point $y \in N$ the vector $\xi(y)$ is not tangent to N. If we assume that the point x is periodic for the flow, that is, $\varphi^{t_0}(x) = x$ for some $t_0 > 0$, then every nearby orbit of the flow intersects the surface N at a time close to t_0 so we have defined for a neighborhood U of x on N a map $F_N: U \to N$ such that $F_N(x) = x$. This map is called a *section map* or *first-return map* or *Poincaré map* for the flow. This construction (also called inducing) also works if x is not periodic but comes sufficiently close to itself (see below).

Finally, for any diffeomorphism $f: M \to M$ one can construct a *suspension flow* on the *suspension manifold* M_f which is obtained from the direct product $M \times [0,1]$ by identifying pairs of points of the form $(x,1)$ and $(f(x),0)$ for $x \in M$. The *suspension flow* σ_f^t is determined by the "vertical" vector field $\frac{\partial}{\partial t}$ on M_f.

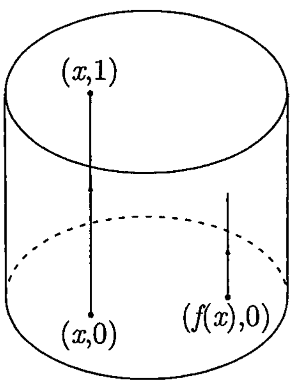

FIGURE 0.3.1. A suspension

This construction is closely related to the solution of a system of ordinary differential equations with periodic coefficients. First recall that a time-dependent system of ordinary differential equations is given by a family of vector fields v_t and thus determines a family of time evolutions $\varphi^{t,s}$ from the moment t to the moment $t+s$ which is not a group. It can, however, be interpreted as a single vector field $w(x,t) = v_t(x) + \dfrac{\partial}{\partial t}$ in the extended phase space $M \times \mathbb{R}$. The time evolution $\Phi^s(x,t) = (\varphi^{t,s}(x), t+s)$ in $M \times \mathbb{R}$ does have the group property. Of course, the space $M \times \mathbb{R}$ is never compact.

The situation changes, however, if the system of ordinary differential equations is periodic in time with period τ, say. Then $v_{t+\tau} = v_t$ and $\varphi^{t+k\tau,s} = \varphi^{t,s}$ for $k \in \mathbb{Z}$. In this case one can reduce the time evolution in $M \times \mathbb{R}$ to one in a factor space by identifying (x,t) with $(x, t+\tau)$. The factor space thus obtained is compact if M is compact and the projection of the flow Φ^s to that space is diffeomorphic to the suspension flow over the map $\varphi^{0,\tau}$ by the map $h\colon (\varphi^{0,t}(x), t) \mapsto (\varphi^{0,\tau}(x), t)$ $(0 \le t \le \tau)$ to $M_{\varphi^{0,\tau}}$.

The suspension construction is generalized to the construction of *the flow under a function* or *special flow*. Namely, add to our data a smooth positive function φ on M and consider the manifold $M_{f,\varphi}$ obtained from the subset $M_\varphi = \{(x,t) \mid x \in M,\ t \in \mathbb{R},\ 0 \le t \le \varphi(x)\}$ of the direct product $M \times \mathbb{R}$ by identifying pairs $(x, \varphi(x))$ and $(f(x), 0)$. Of course, topologically $M_{f,\varphi}$ is the same as M_f, but the "vertical" vector field $\dfrac{\partial}{\partial t}$ on $M_{f,\varphi}$ determines a new flow $\sigma^t_{f,\varphi}$ (the special flow under φ built over f) which differs from the suspension by a time change (see Definition 2.2.3).

Exercises

0.3.1. *Let $M = [0,1]$ and $f(x) = 1 - x$. Show that the manifold M_f is homeomorphic to the Möbius strip. The suspension flow has one orbit of period one and infinitely many orbits of period two. Show that the period-one orbit does not separate M_f and that any period-two orbit except the one that forms the boundary separates it into two pieces, one homeomorphic to the Möbius strip and the other to the cylinder $[0,1] \times S^1$.*

0.3.2. *Let $M = S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and $f(z) = -z$. Show that the manifold M_f is homeomorphic to the two-torus $\mathbb{T}^2 = S^1 \times S^1$.*

0.3.3. *Let $M = S^1$ and $f(z) = \bar{z}$. Show that the manifold M_f is homeomorphic to the Klein bottle. The suspension flow has two orbits of period one and infinitely many orbits of period two. Show that period-one orbits do not separate M_f and that any period-two orbit separates it into two pieces homeomorphic to the Möbius strip.*

0.3.4. *Describe the smooth structure on the suspension manifold M_f and, more generally, on the manifold $M_{f,\varphi}$.*

4. Linearization and localization

We will see in the next three chapters that a large number of useful concepts related to the asymptotic behavior of smooth dynamical systems in fact belong to topological dynamics, that is, they are defined only in terms of topology, not the differentiable structure. We already mentioned some reasons for that in Section 0.1. Now we would like to point out some specific features that distinguish the theory of smooth dynamical systems from topological dynamics.

Already in elementary calculus one learns how useful it is to represent a function $\varphi(t)$ of one real variable t near a point t_0 as the main linear part $\varphi(t_0)+\varphi'(t_0)(t-t_0)$ plus an "infinitesimal of higher order", $o(t-t_0)$. A less elementary version of the same idea plays a central role in the theory of smooth dynamical systems. First, if $U\subset\mathbb{R}^m$ is an open neighborhood of x_0 and $f\colon U\to\mathbb{R}^m$ is a differentiable map, we can represent f near the point x_0 as the constant part $f(x_0)$ plus the linear part $Df_{x_0}(x-x_0)$ plus higher-order terms. The differential Df is a linear operator in $\mathbb{R}^n$ that is represented in coordinate form by the matrix of partial derivatives. If $f(t_1,\dots,t_m)=(f_1(t_1,\dots,t_m),\dots,f_m(t_1,\dots,t_m))$, then

$$Df_{x_0}(t_1,\dots,t_m)=\left(\frac{\partial f_i}{\partial t_j}\right)_{i,j=1,\dots,m},$$

where the partial derivatives are calculated at the values of the coordinates corresponding to the point x_0. If the map is regular at x_0 this operator is invertible.

The picture remains essentially the same for differentiable maps of smooth manifolds with the only difference that instead of the standard coordinate system in $\mathbb{R}^m$ one should use appropriate local coordinate systems near a point and its image. A more invariant way to express the same idea is to describe the differential Df_{x_0} of the map $f\colon M\to M$ as a linear map of the tangent space $T_{x_0}M$ into the space $T_{f(x_0)}M$. If f is a diffeomorphism then the differential is invertible. This construction can be globalized by considering the tangent bundle $TM=\bigcup_{x\in M}T_xM$ which can be provided with the structure of a differentiable manifold whose dimension is twice the dimension of M (see Section 3 of the Appendix). Any local coordinate system on M induces a coordinate system in TM which is global in the tangent direction. Namely, tangent vectors to the coordinate curves form a basis in each tangent space and the $2n$ coordinates of a tangent vector include n coordinates of its base point plus the coordinates of the vector with respect to that basis.

When we consider iterates of a map f, the differential $Df^n_x\colon T_xM\to T_{f^n_x}M$ of the nth iterate is a composition of the differentials $Df_{f^i(x)}\colon T_{f^i(x)}\to T_{f^{i+1}(x)}$, $i=0,\dots,n-1$:

$$\underbrace{T_xM\xrightarrow{Df_x}T_{f(x)}M\xrightarrow{Df_{f(x)}}T_{f^2(x)}M\xrightarrow{Df_{f^2(x)}}\dots\xrightarrow{Df_{f^{n-1}(x)}}T_{f^n(x)}M}_{Df^n_x}.$$