Introduction

Perturbation of the boundary (or of the domain of definition of a boundary value problem) is a rather neglected mathematical topic, though it has attracted occasional interest: Rayleigh [33] in 1877 (the first edition), Hadamard [9] in 1908, Courant and Hilbert [5] in the German edition of 1937, Polya and Szëgo [31] in 1951, Garabedian and Schiffer [7] in 1952, and some more recent work [3, 14, 6, 13, 26, 21–23, 42, 35, 32, 27, 10, 11]. The list is far from complete, but is notably sparse.

There seem to be two related reasons for this neglect: (1) the subject is too easy; (2) it is too difficult. If you are interested only in Fundamental Questions, this is certainly a trivial topic. One perturbs the region by applying a diffeomorphism near the identity; but you can change variables via this diffeomorphism to keep the region fixed, and are then only perturbing the coefficients in a fixed region. It is simply the chain rule. However, if you try to carry out this trivial change of variables, you will become mired in such long and difficult calculations that you'll be tempted to quit. If you persist, and are fortunate, and are extremely careful, there may be a miraculous simplification at the end. On experiencing this miracle for the second time I became suspicious – theorems 2.2, 2.4 show how to go directly to the "miracle," bypassing the computational morass. (Peetre [27] also found part of this result, and it is implicit in Courant and Hilbert [5: vol. 1 p. 260] for variational problems.) It is, at the end, merely the chain rule, and we may then apply standard tools (implicit function theorem, Liapunov, Schmidt method, transversality theorem) to problems of perturbation of the boundary. But the standard tools are not enough – new problems arise requiring, for example, a more general form of the transversality theorem for problems with Fredholm index $-\infty$ (ch. 5). Open problems abound. The calculus developed in chapter 2 applies – in principle – to almost any boundary or initial boundary-value problem, but often leads immediately to difficult unsolved problems. To avoid excessive depression of author and reader, we will concentrate on questions we can answer – generally boundary-value problems for scalar second-order elliptic equations. The subject is not, after all, entirely trivial.

I have worked on perturbation of the boundary sporadically since about 1973, after reading Joseph's article [13]. The formulas of theorems 2.2, 2.4 date from 1975 – in a more complicated version – and most of the examples of chapters 3 and 4 (and a few from chapter 6) were developed in 1975–1981 and exposed in seminars at the University of Kentucky, Brown University and the University of São Paulo. In 1982, I had the opportunity to develop this topic at some length [11] at the University of Brasilia. Later that year I generalized the transversality theorem (lectures at São Carlos, September 1982), which solved some problems left open in [11] and raised a host of new open problems. These demanded some tedious calculation – such as those in chapter 7 – which were only completed recently, and then only in special cases: much remains to be done. If the word "I" seems to appear excessively here, it is because very few other people have worked on these problems in the past twelve years with a comparable approach, and my work has been independent of these few (aside from some of the examples cited below). There are, of course, other notions of a "small change in the domain" besides "image under diffeomorphism near the identity" (see, for example, [3, 26, 10]); the advantage of using such regular perturbations will hopefully become more clear as we proceed.

In the past five years, my work has been supported by FAPESP, and of course by IME-USP. It would have been nice to have this ready for the fiftieth anniversary of the University of São Paulo, but as usual I missed the deadline. Still, better late than never: Happy 51st birthday!

Most of the following (Chapters 1 to 7) was written in 1985. But at the end of that year, I found a way to circumvent Horrible Chapter Seven, the method of rapidly oscillating solutions. My course was clear: everything should be rewritten with the new method! Unfortunately, I couldn't seem to find the time and energy needed for the task. Finally, Jack Hale and Antonio Luiz Pereira persuaded me to publish it as it stands, with only minor corrections. (Except that the correction of Theorem 7.6.10 is not so minor, and this required rewriting Example 6.8.) A new chapter was added, on the method of rapidly oscillating solutions, with a new example (8.5: generic simplicity of solutions of a system) to show the power of the method. And, with a few years perspective, Chapter 7 does not seem so horrible.

Thanks Jack and Antonio Luiz for getting it moving!

Chapter 1

Geometrical Preliminaries

In this chapter we introduce some notation, find convenient representations for regions in $\mathbb{R}^n$ with smooth boundary (1.2–1.7) and develop some tools needed later: approximation by smooth regions (1.8), smooth extension of functions (1.9), smooth deformations of regions (1.10), differentiation of integrals (1.11), and differential operators on a hypersurface (1.12). In short, a mixed bag of topics which will be needed later, and it may be wise to omit details until the need becomes apparent.

We treat only smoothly-bounded regions. Initial-boundary-value problems lead naturally to the study of regions with corners; but our examples will be elliptic boundary-value problems, and smooth regions provide sufficient variety.

1.1 Some Notation

For a function f defined near $x \in \mathbb{R}^n$, the m^{th} derivative at x, $D^m f(x)$, may be considered as a homogeneous polynomial of degree m ($h \mapsto D^m f(x)h^m$) on $\mathbb{R}^n$, or as a symmetric m-linear form, or as the collection of partial derivatives

$$D^m f(x) = \left\{ \left(\frac{\partial}{\partial x} \right)^\alpha f(x) : |\alpha| = m \right\},$$

depending on convenience. Then the norm $|D^m f(x)|$ may denote

$$\max_{|\alpha|=m} \left| \left(\frac{\partial}{\partial x} \right)^\alpha f(x) \right| \text{ or } \max_{|h|\le 1} |D^m f(x)h^m|.$$

The last version has the advantage of being independent of rotation of coordinates in $\mathbb{R}^n$, but the norms are equivalent.

If Ω is an open set in $\mathbb{R}^n$ and $m \ge 0$ an integer, $C^m(\Omega)$ is the space of m-times continuously and bounded differentiable functions on Ω whose derivatives

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extend continuously to the closure $\overline{\Omega}$, with the usual norm

$$\|\phi\|_{C^m(\Omega)} = \max_{0\le j\le m} \sup_{x\in\Omega} |D^j\phi(x)|.$$

The space of values is some normed linear space E, and is not clear which "E" is meant, we may write $C^m(\Omega, E)$.

- $C^m_{unif}(\Omega)$ is the closed surface of $C^m(\Omega)$ consisting of functions whose m^{th} derivative is uniformly continuous; if Ω is bounded, this is $C^m(\Omega)$.
- $C^{m,\alpha}(\Omega)$ is the subspace of $C^m_{unif}(\Omega)$ consisting of functions whose m^{th} derivative is Hölder continuous with exponent α ($0 < \alpha \le 1$), provided with the norm

$$\|\phi\|_{C^{m,\alpha}(\Omega)} = \max\left(\|\phi\|_{C^m(\Omega)}, H^{\Omega}_{\alpha}(D^m\phi)\right)$$

  where

$$H^{\Omega}_{\alpha}(f) = \sup\{|f(x) - f(y)|/|x-y|^{\alpha} : x \ne y \in \Omega\}$$

  (This space is "boundedly closed" in $C^0(\Omega)$; that is, a bounded sequence in $C^{m,\alpha}(\Omega)$ which converges uniformly (in $C^0(\Omega)$) has its limits in $C^{m,\alpha}(\Omega)$.)
- $C^{m,\alpha+}(\Omega)$ is the closed subspace of $C^{m,\alpha}(\Omega)$, $0 < \alpha < 1$, consisting of functions $\phi \in C^{m,\alpha}(\Omega)$ such that

$$|D^m\phi(x) - D^m\phi(y)|/|x-y|^{\alpha} \to 0 \quad \text{as } x - y \to 0 \quad (x, y \in \Omega),$$

  provided with the same $C^{m,\alpha}(\Omega)$ norm.

It is sometimes convenient to write

$$C^{m,0} \text{ for } C^m, \qquad C^{m,0+} \text{ for } C^m_{unif}$$

so we allow $0 \le \alpha \le 1$ in $C^{m,\alpha}$ and $0 \le \alpha < 1$ in $C^{m,\alpha+}$. $C^{m,\alpha}_{loc}(\Omega)[C^{m,\alpha+}_{loc}(\Omega)]$ is the space of functions whose restrictions to any Ω_0 (with $\overline{\Omega}_0$ compact in Ω) are in $C^{m,\alpha}(\Omega_0)[C^{m,\alpha+}(\Omega_0)$, respectively].

$$C^{\infty}(\Omega) = \bigcap_m C^m(\Omega), \qquad C^{\infty}_{loc}(\Omega) = \bigcap_m C^{\infty}_{loc}(\Omega)$$

$C^{\omega}(\Omega)$ = analytic functions defined on an open set of $\mathbb{R}^n$ containing $\overline{\Omega}$ (hence, extending to analytic functions on an open set in $\mathbb{C}^n$, when we consider $\mathbb{R}^n$ as $\{z \in \mathbb{C}^n : \text{Im } z = 0\}$.)

Definition 1.2. An open set $\Omega \subset \mathbb{R}^n$ has C^m*-regular boundary* [or $C^{m,\alpha}$ or $C^{m,\alpha+}$ or $C^{m,\alpha}_{loc}$ or $C^{m,\alpha+}_{loc}$ or C^{∞} or C^{∞}_{loc} or C^{ω}; regular boundary], if there exists $\phi \in C^m(\mathbb{R}^n, \mathbb{R})$ [or $C^{m,\alpha}$ or $C^{m,\alpha+}$ or ...] which is at least in $C^1_{unif}(\mathbb{R}^n, \mathbb{R})$ such that

$$\Omega = \{x \in \mathbb{R}^n : \phi(x) > 0\}$$

and $\phi(x) = 0$ implies $|\text{grad}\phi(x)| \geq 1$. We may also say Ω is a C^m region or Ω is C^m *regular* [or $C^{m,\alpha}$ or ...].

Theorem 1.3. *$\Omega \subset \mathbb{R}^n$ has C^m-regular boundary (or $C^{m,\alpha}$ or $C^{m,\alpha+}$) – at least C^1_{unif} – if and only if there are positive constants r, M such that, given any open ball $B \subset \mathbb{R}^n$ of radius r, after appropriate rotation and translation of coordinates, we have*

$$\Omega \cap B = \{x \in \mathbb{R}^n | x_n > \psi(\hat{x})\} \cap B, \quad \hat{x} = (x_1, \ldots, x_{n-1})$$
$$\partial\Omega \cap B = \{x \in \mathbb{R}^n | x_n = \psi(\hat{x})\} \cap B$$

for some $\psi \in C^m(\mathbb{R}^{n-1}, \mathbb{R})$ (or $C^{m,\alpha}$ or $C^{m,\alpha+}$) with norm $\leq M$. Note the conditions are trivial if $B \cap \partial\Omega = \emptyset$.

Remark. This implies easily that our definition of C^m-regular boundary is equivalent to that used by F. Browder and Agmon-Douglis-Nirenberg in their studies of elliptic boundary value problems. We will see that Def. 1.2 is very convenient for discussing perturbations of the boundary (as in 1.8 below). Our Definition 1.2 applied only when $\partial\Omega$ is at least uniformly C^1. The condition of Thm. 1.3 is more general, in that we may permit (for example) ψ to be merely Lipschitz continuous (in $C^{0,1}(\mathbb{R}^{n-1})$) which gives the "minimally smooth" domains of Stein's extension theorem [38, Sec. 6.3]. Some of our results apply to regions with convex corners, transversal intersections of smooth regions, as described in the remark following Thm. 1.9.

Proof. Suppose $\Omega = \{x \,|\, \phi(x) > 0\}$ is C^m (or $C^{m,\alpha}$ or $C^{m,\alpha+}$) regular, $\phi(x) = 0 \Rightarrow |\text{grad}\,\phi(x)| \geq 1$, $L = \sup|\text{grad}\,\phi|$, and choose $r > 0$ so $|x - y| \leq 6Lr \Rightarrow |D\phi(x) - D\phi(y)| < 1/2$.

Let B be a ball of radius r in $\mathbb{R}^n$ which meets $\partial\Omega$; we may assume the center of the ball is 0. If $0 \in \partial\Omega$, choose the positive x_n-axis along the inward normal (i.e., grad ϕ (0)). Otherwise let p be a point of $\partial\Omega \cap B$ closest to 0 and choose the x_n- axis to contain p and be directed into Ω. Then $p = (0, p_n)$, $|p_n| < r$, and $\frac{\partial\phi}{\partial x_n}(p) = |D\phi(p)| \geq 1$ (possibly $p = 0$). Also $\phi\,(0, s)$ has the same sign as $s - p_n$ in $-r < s < r$, and for $|x - p| \leq 6Lr$ we have $\frac{\partial\phi}{\partial x_n}(x) > 1/2$.

Let $\hat{x} \in \mathbb{R}^{n-1}$, $|\hat{x}| \leq 2r$; then $|\phi(\hat{x}, x_n) - \phi(0, x_n)| \leq 2Lr$ and $\pm\phi(\hat{x}, p_n \pm 4Lr) \geq \pm\phi(0, p_n \pm 4Lr) - 2Lr > 0$ so there exists unique $\psi(\hat{x}) \in (p_n - 4Lr, p_n + 4Lr)$ with $\phi(\hat{x}, \psi(\hat{x})) = 0$. By the implicit function theorem, ψ is C^m (or $C^{m,a}$ or $C^{m,\alpha+}$, respectively) and $|D\psi(\hat{x})| = |-(\partial\phi/\partial\hat{x})/(\partial\phi/\partial x_n)| \leq 2M$ for $|\hat{x}| \leq 2r$. Choose some (fixed) $C^\infty\theta : \mathbb{R} \to [0, 1]$ with $\theta(t) = 1$ for $t \leq 1$, $\theta(t) = 0$ for $t \geq 3/2$, and let $\psi_0(\hat{x}) = \theta(|\hat{x}|/r)\psi(\hat{x})$, or zero for $|\hat{x}| \geq 3r/2$. Then $\psi_0 \in C^m(\mathbb{R}^{n-1})$ (or $C^{m,\alpha}$ or $C^{m,\alpha+}$) with norm bounded by a multiple (depending only on n, m, α, r) of the norm of ϕ, and

$$\Omega \cap B = \{x \in B \mid x_n > \psi_0(\hat{x})\}, \quad \partial\Omega \cap B = \{x \in B \mid x_n = \psi(\hat{x})\}.$$

For the converse, we need the following

Lemma 1.4. *Let $\sigma > \sqrt{n}, r > 0$, and for each $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$, let B_k be the open ball in $\mathbb{R}^n$ with radius r and center $(r/\sigma)k$, while $B_k^{1/2}$ is the concentric ball with radius $r/2$. Then every point of $\mathbb{R}^n$ is contained in some $B_k^{1/2}$, and no point is contained in more than $(2\sigma + 1)^n$ of the balls B_k.*

*Proof of the Lemma***.** The result is invariant under a homothety ($x \mapsto cx, c =$ constant > 0) so it suffices to treat the case $r = \sigma$. If $x \in \mathbb{R}^n$ there exists $k \in \mathbb{Z}^n$ so $x - k \in [-\frac{1}{2}, \frac{1}{2})^n$, hence $|x - k| \le \sqrt{n}/2 < \sigma/2$ so $x \in B_k^{1/2}$.

Suppose $x \in B_k$; then for each $j = 1, \dots n, |x_j - k_j| < \sigma$ so k_j is an integer in $(x_j - \sigma, x_j + \sigma)$. But $(x_j - \sigma, x_j + \sigma)$ contains no more than $2\sigma + 1$ integers, so there are at most $(2\sigma + 1)^n$ choices of $k \in \mathbb{Z}^n$ such that $x \in B^k$.

Completion of Proof of (1.3). Assume r, M given satisfying the requirements of the theorem; we must find $\phi : \mathbb{R}^n \to \mathbb{R}$ which satisfies the conditions (1.2). Choose $\sigma = \sqrt{2n + 1}$ in the lemma. There is a C^∞ partition of unity $\{\phi_k\}_{k \in \mathbb{Z}^n}$

$$\text{supp } \phi_k \subset B_k^{1/2}, \quad \phi_k \ge 0, \quad \sum_k \phi_k(x) \equiv 1$$

with $\|\phi_k\|C^m$ (or $\|\phi_k\|C^{m,\alpha}$) uniformly $\le K = K(n, r, m, \alpha)$.

If $B_k \cap \partial\Omega \ne \phi$, there is a function $S_k(x)$ of class C^m(or $C^{m,\alpha}$or $C^{m,\alpha+}$) – $S_k(x) = x_n - \psi(\hat{x})$ after rotation of coordinates – such that

$$\Omega \cap B_k = \{x \in B_k : S_k(x) > 0\},$$
$$\partial\Omega \cap B_k = \{x \in B_k : S_k(x) = 0\},$$

with $\|S_k\|_{C^m \text{ (or } C^{m,\alpha})} \le M$ and

$$S_k(x) = 0 \Rightarrow |DS_k(x)| \ge 1.$$

If $B_k \subset \Omega$, let $S_k = 1$; if $B_k \cap \overline{\Omega} = \emptyset$, let $S_k = -1$. Define $\psi(x) = \sum_k \phi_k(x) S_k(x)$. Then ψ is in $C^m(\mathbb{R}^n, \mathbb{R})$ [or $C^{m,\alpha}$or $C^{m,\alpha+}$], $\psi > 0$ in Ω, $\psi = 0$ on $\partial\Omega$, $\psi < 0$ outside $\overline{\Omega}$. If $x \in \partial\Omega$ then $\phi_k(x)S_k(x) = 0$ for each k and at x, grad $(\phi_k S_k) = \phi_k$ grad S_k which either vanishes or has the direction of the inward normal so $|\text{grad } \psi(x)| = \sum_k \phi_k(x)|\text{ grad } S_k(x)| \ge 1$.

The following “normal coordinates” are sometimes useful.

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^n$ have C^m-regular boundary (or $C^{m,\alpha}$ or $C^{m,\alpha+}$, $2 \le m \le \infty$). There exists $r > 0$ so that if*

$$B_r(\partial\Omega) = \{x : \text{dist}(x, \partial\Omega) < r\}$$
$$\pi(x) = \textit{the point of } \partial\Omega \textit{ nearest to } x$$
$$t(x) = \pm\text{dist}(x, \partial\Omega) \quad (\text{“}+\text{” } \textit{outside}, \text{“}-\text{” } \textit{inside})$$

then $t(\cdot) : B_r(\partial\Omega) \to (-r, r)$, $\pi(\cdot) : B_r(\partial\Omega) \to \partial\Omega$ are well-defined, π is a C^{m-1} (or $C^{m-1,\alpha}$ or $C^{m-1,\alpha+}$) retraction onto $\partial\Omega$ ($\pi(x) = x$ when $x \in \partial\Omega$) and t has the same smoothness as $\partial\Omega$ (C^m or $C^{m,\alpha}$ or $C^{m,\alpha+}$). Further

$$x \mapsto (t(x), \pi(x)) : B_r(\partial\Omega) \to (-r, r) \times \partial\Omega$$

is a C^{m-1} (or $C^{m-1,\alpha}$ or $C^{m-1,\alpha+}$) diffeomorphism with inverse

$$(t, \xi) \mapsto \xi + tN(\xi) : (-r, r) \times \partial\Omega \to B_r(\partial\Omega)$$

where $N(\xi)$ is the unit outward normal to $\partial\Omega$ at ξ.

$$t(\cdot) \textit{ is the unique solution of } |\nabla t(x)| = 1 \textit{ in } B_r(\partial\Omega),$$

with $t = 0$ on $\partial\Omega$, $\partial t/\partial N > 0$ on $\partial\Omega$.

Extending the normal field N to a neighborhood of $\partial\Omega$ by

$$N(\xi + tN(\xi)) = N(\xi) \quad -r < t < r,$$

we have $N(x) = \text{grad}\, t(x)$ on $B_r(\partial\Omega)$. Also $K(x) = DN(x) = D^2t(x)$, restricted to the tangent space at $x \in \partial\Omega$, is the curvature of $\partial\Omega$. It is sometimes convenient to call $K(x)$ the curvature, though it is degenerate ($K(x)N(x) = 0$) in the normal direction.

Remark. The fact that $t(\cdot)$ has the same smoothness as $\partial\Omega$ – does not lose a derivative, as happens with $\pi(\cdot)$ – seems to have been noted first by Gilbarg and Trudinger [8].

The best (largest) choice of r is $r = 1/\max|k|$, where k is the sectional curvature of the boundary in any (tangent) direction at any point of $\partial\Omega$.

Corollary 1.6. *A C^m-regular region $\Omega \subset \mathbb{R}^n$, $m \ge 2$, may be represented by $\{x | \phi(x) > 0\}$ where ϕ is C^m and $|\nabla\phi(x)| \equiv 1$ on a neighborhood of $\partial\Omega$. In this case, ϕ is unique on a neighborhood of $\partial\Omega$.*

Proof of (1.5). By the inverse function theorem, the C^{m-1} map $(t, \xi) \mapsto \xi + tN(\xi) = x$ has a C^{m-1} inverse $t = t(x)$, $\xi = \pi(x)$, on some neighborhood of $\partial\Omega$. On the other hand, for each $x \in \mathbb{R}^n$, $\xi \mapsto \frac{1}{2}|x - \xi|^2$ ($\xi \in \partial\Omega$) has a minimum and if ξ is a minimizing point then $x - \xi \perp T_\xi(\partial\Omega)$ so $x = \xi + tN(\xi)$

for some real t with $t = \pm\text{dist}(x, \partial\Omega)$. We show, for some $r > 0$, that $x = \xi + tN(\xi)$, $t = \pm\text{dist}(x, \partial\Omega)$ has a unique solution ξ whenever $x \in B_r(\partial\Omega)$ so $\xi = \pi(x)$ is the (unique) nearest points.

Suppose $\Omega = \{x : \phi(x) > 0\}$, $\phi(x) = 0 \Rightarrow |\nabla\phi(x)| \geq 1$ and $r = 1/\sup|D^2\phi(x)|$. If $\text{dist}(x, \partial\Omega) < r$ but there are two "nearest points" $\xi_1 \neq \xi_2$, $x = \xi_j + tN(\xi_j)$, then $t^2 = |\xi_1 - \xi_2 + tN(\xi_1)|^2 = |\xi_1 - \xi_2|^2 + 2t(\xi_1 - \xi_2)$. $N(\xi_1) + t^2$. Now $\phi(\xi_2) = \phi(\xi_1) = 0 = \nabla\phi(\xi_1)\cdot(\xi_2 - \xi_1) + ½\int_0^1 D^2\phi(\theta\xi_2 + (1-\theta\xi_1))\cdot(\xi_2 - \xi_1)^2 d\theta$ and $N(\xi_1) = -\nabla\phi(\xi_1)/|\nabla\phi(\xi_1)|$, so for $|t| < r$

$$0 < |\xi_1 - \xi_2|^2 = 2tN(\xi_1)\cdot(\xi_2 - \xi_1) \leq |t|\sup|D^2\phi|\,|\xi_1 - \xi_2|^2 < |\xi_1 - \xi_2|^2,$$

a contradiction. Thus $t(\cdot)$, $\pi(\cdot)$ are well-defined and C^{m-1} on $B_r(\partial\Omega)$.

Extend N to be constant on normal lines, so $N(x) = N(\pi(x))$ is C^{m-1} on $B_r(\partial\Omega)$. It is clear that $t(x + sN(x)) = t(x) + s$ when $t(x)$ and $t(x) + s$ are in $(-r, r)$, so

$$\partial t(x)/\partial x_j = \partial_j t(x + sN(x)) + s\sum_{k=1}^{n}\partial_k t(x + sN(x))\partial_j N_k(x).$$

Let $s = -t(x)$, so $x + sN(x) = \pi(x) \in \partial\Omega$, and note $\nabla t(\pi(x)) = N(\pi(x)) = N(x)$. Then on $B_r(\partial\Omega)$

$$\partial_j t(x) = N_j(x) - t(x)\sum_{k=1}^{m} N_k(x)\partial_j N_k(x) = N_j(x)$$

or $N(x) = \text{grad}\, t(x)$. (Observe that $K(x)N(x) = \partial N/\partial N(x) = 0$.) Clearly N is C^{m-1} so $t(\cdot)$ is C^m.

To identify $DN(\xi) = D^2t(\xi)$ as the curvature of $\partial\Omega$ at ξ, we may choose coordinates so Ω is locally $\{x_n > \psi(\hat{x})\}$ where $\psi(0) = 0$, $D\psi(0) = 0$ and $K_{ij} = \partial^2\psi/\partial x_i\partial x_j\,(0)\,(1 \leq i, j \leq n-1)$ is the curvature matrix (on the tangent plane $\mathbb{R}^{n-1} \times 0$). It is easy to show $N(x) = (0, -1) + (K\hat{x}, 0) + o(|x|)$ as $x \to 0$ so $DN(0) = \left(\begin{smallmatrix} K & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and the restriction in the tangent plane is the curvature K of $\partial\Omega$ at 0.

Remark. If S is a C^2 hypersurface, $x_0 \in S$, the usual existence theory [5], [6] for first order scalar PDEs says there is a unique C^1 solution of $|\nabla t| = 1$ near x_0 with $t = 0$, $\partial t/\partial N > 0$ on S near x_0. If S is not C^2 (or $C^{1,1}$) there may be no C^1 solution of this problem. For example if $1 < p < 2$, $S = \{x : x_2 = |x_1|p\}$, there are two "nearest points" to x when $x_1 = 0$, $x_2 > 0$ (near 0) and the gradient of $\text{dist}(x, S)$ has a discontinuous jump as x_1 crosses 0 with $x_2 > 0$.

Theorem 1.7. *Let Γ be a compact subgroup of the orthogonal group $O(n)$ and let $\Omega \subset \mathbb{R}^n$ be a C^m-regular region such that $\gamma(\Omega) = \Omega$ for all $\gamma \in \Gamma$. Then there exists C^m $\phi : \mathbb{R}^n \to \mathbb{R}$ such that for $\Omega = \{x : \phi(x) > 0\}$,*

$\phi(x) = 0 \Rightarrow |\nabla\phi(x)| \geq 1$, *and* $\phi(\gamma \cdot x) = \phi(x)$ *for all* $\gamma \in \Gamma, x \in \mathbb{R}^n$. *If* $\partial\Omega$ *is at least* C^2, *we may choose such* ϕ *with* $|\nabla\phi| = 1$ *on a neighborhood of* $\partial\Omega$.

Proof. There exists a function ϕ_0 satisfying all requirements except perhaps Γ-invariance. Let $\phi(x)$ be the average of $\gamma \mapsto \phi_0(\gamma x)$ with respect to Haar measure in Γ; then ϕ is certainly C^m and Γ-invariant. Further $x \in \Omega \Rightarrow \gamma x \in \Omega \Rightarrow \phi_0(\gamma x) > 0$ for all $\gamma \in \Gamma$ so $\phi(x) > 0$ in Ω; and similarly $\phi(x) = 0$ on $\partial\Omega$, $\phi(x) < 0$ outside $\overline{\Omega}$. On $\partial\Omega$, $N(x) = -\nabla\phi_0(x)/|\nabla\phi_0(x)|$ and it follows easily that $N(\gamma x) = \gamma N(x)$ for $x \in \partial\Omega$, $\gamma \in \Gamma$, so

$$\text{grad}(\phi_0(\gamma x)) = \gamma(\text{grad } \phi_0(\gamma x) = -N(x)|\text{grad } \phi_0(\gamma x)|$$

and (averaging with respect to γ) $|\text{grad } \phi(x)| = \text{average}_\gamma |\text{grad } \phi_0(\gamma x)| \geq 1$ for $x \in \partial\Omega$. If in fact $|\nabla\phi_0(x)| = 1$ near $\partial\Omega$, then $\phi(x) = \phi_0(x) = \pm\text{dist}(x, \partial\Omega)$ near $\partial\Omega$ so $|\nabla\phi(x)| = 1$ near $\partial\Omega$.

A common technique in analysis is to approximate a given function by a smooth function, to facilitate calculations, and take limits only at the end. Similarly we may approximate a given region by smooth regions. If $\Omega = \{x : \phi(x) > 0\}$ is a C^m_{unif} region, and ψ is C^m-close to ϕ, we show $\{x : \psi(x) > 0\}$ is C^m-close to Ω, that is, it is diffeomorphic to Ω by a diffeomorphism C^m-close to the identity.

Theorem 1.8. *Let* $\phi : \mathbb{R}^n \to \mathbb{R}$ *be (at least) uniformly* $C^1, \phi(x) = 0 \Rightarrow |\text{grad } \phi(x)| \geq 1$, $\Omega = \{x : \phi(x) > 0\}$, *and for some* $a > 0$,

$$|\phi(x)| \geq \min\left\{\frac{1}{2}\,\text{dist}(x, \partial\Omega), a\right\} \quad \textit{for all } x.$$

The last condition may always be achieved by modifying ϕ *away from* $\partial\Omega$. *Also, let* $r_0 > 0$.

Then there exists $\epsilon_0 > 0$ such that, if $\|\psi - \phi\|_{C^1(\mathbb{R}^n)} < \epsilon_0$, there is a diffeomorphism $h(\cdot, \psi) : \mathbb{R}^n \to \mathbb{R}^n$ supported in $B_{r_0}(\partial\Omega)$ – i.e., $h(x, \psi) = x$ when dist $(x, \partial\Omega) \geq r_0$ – such that $h(\Omega; \psi) = \{x : \psi(x) > 0\}$ and $\|h(\cdot, \psi) - id_{\mathbb{R}^n}\|_{C^1(\mathbb{R}^n)} \to 0$ as $\|\psi - \phi\|_{C^1(\mathbb{R}^n)} \to 0$. If ψ is of class $C^{m,\alpha}$ [or $C^{m,\alpha+}$ or C^∞ or C^ω] then $\{x : \psi(x) > 0\}$ is a $C^{m,\alpha}$ [or $C^{m,\alpha+}$ or C^∞ or C^ω] regular region, assuming $\|\psi - \phi\|_{C^1} < \epsilon_0$.

If ϕ, ψ are $C^{m,\alpha+}$ then $h(\cdot, \psi)$ is a $C^{m,\alpha+}$ diffeomorphism, $\|h(\cdot, \psi) - id_{\mathbb{R}^n}\|_{C^{m,\alpha}} \to 0$ as $\|\psi - \phi\|_{C^{m,\alpha}} \to 0$ and $(x, \psi) \mapsto h(x, \psi) : \mathbb{R}^n \times C^{m,\alpha+}(\mathbb{R}^n) \to \mathbb{R}^n$ is $C^{m,\alpha+}$.

If ϕ, ψ are $C^{m,\alpha}$ then $h(\cdot, \psi)$ is a $C^{m,\alpha}$ diffeomorphism and $(x, \psi) \mapsto h(x, \psi) : \mathbb{R}^n \times C^{m,\alpha} \to \mathbb{R}^n$ is $C^{m,\alpha}$. As $\|\psi - \phi\|_{C^1} \to 0$ with $\|\psi\|_{C^{m,\alpha}}$ bounded, $h(\cdot, \psi)$ remains bounded in $C^{m,\alpha}$ and converges to $id_{\mathbb{R}^n}$ in $C^{k,\beta}$ when $k + \beta < m + \alpha$.

Remark. The hypothesis $\|\psi - \phi\|_{C^{m,\alpha}} \to 0$ does not yield $C^{m,\alpha}$ convergence of $h(\cdot, \psi) \to id$, if ϕ is not $C^{m,\alpha+}$.

Proof. Choose a C^∞ vector field $M(x)$ on a neighborhood of $\partial\Omega$, uniformly close to grad ϕ: for some $r_1 > 0$ and $C \geq 2$

$$\frac{1}{2} \leq |M(x)| \leq C$$

and

$$M(x) \cdot \text{grad}\, \phi(x) \geq \frac{1}{2}|M(x)|^2 \quad \text{if dist}(x, \partial\Omega) \leq r_1.$$

Let $r_2 = \min\{a, r_0, \tfrac{1}{2}r_1\}$ and choose C^∞ $\theta : \mathbb{R}^n \to [0, 1]$ so that $\theta = 1$ on $B_{r_2/2}(\partial\Omega)$, $\theta = 0$ outside $B_{r_2}(\partial\Omega)$. We will show that, if $\|\psi - \phi\|_{C^1}$ is small and $x \in B_{r_2}(\partial\Omega)$, there is a unique $t = t(x; \psi)$ near 0 such that

$$\psi(x + tM(x)) = \phi(x).$$

Then define h by

$$h(x; \psi) = \begin{cases} x + \theta(x)t(x; \psi) & M(x) \text{ in } B_{r_2}(\partial\Omega) \\ x & \text{outside } B_{r_2}(\partial\Omega) \end{cases}$$

and the conditions of the theorem are satisfied.

First choose $s_0 > 0$ so small that $s_0 \leq \tfrac{1}{2}r_2$ and $|D\phi(x) - D\phi(y)| \leq 1/8C$ when $|x - y| \leq CS_0$. Suppose $\|\psi - \phi\|_{C^0} = \sup|\psi - \phi| \leq s_0/32$ and $\sup|D\psi - D\phi| \leq 1/8C$. Then for $-s_0 \leq t \leq s_0$, dist$(x, \partial\Omega) \leq r_2$,

$$\begin{aligned} \frac{\partial}{\partial t}(\psi(x + tM(x)) - \phi(x)) &= M(x) \cdot D\psi(x + tM(x)) \\ &= M(x) \cdot D\phi(x) + M(x) \cdot (D\psi(x + tM(x)) - D\phi(x)) \\ &\geq \frac{1}{2}|M|^2 - |M|\left(\frac{1}{8C} + \frac{1}{8C}\right) \geq \frac{1}{16} \end{aligned}$$

while for $t = \pm s_0$

$$\begin{aligned} t(\psi(x + tM(x)) - \phi(x)) &\geq t(\phi(x + tM(x)) - \phi(x)) - s_0\|\psi - \phi\|_{C^0} \\ &\geq s_0^2\left(\frac{1}{2}|M|^2 - |M|/8c\right) - \frac{s_0^2}{32} \geq \frac{s_0^2}{32} > 0. \end{aligned}$$

Thus there is a unique $t = t(x, \psi)$ in $(-s_0, s_0)$ with $\psi(x + tM(x)) = \phi(x)$, and it is easy to see

$$|t(x, \psi)| \leq 16\|\psi - \phi\|_{C^0} \quad \text{and} \quad \frac{\partial}{\partial x}t(x; \psi) \to 0$$

uniformly on $B_{r_2}(\partial\Omega)$ as $\|\psi - \phi\|_{C^1} \to 0$. (Here we again use the uniform continuity of $D\psi$.)