NUMERICAL METHODS IN FINANCE

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CONTENTS

Convergence of Numerical Schemes for Degenerate Parabolic Equations Arising in Finance Theory

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1 Introduction

The aim of this article is twofold: on one hand, we describe a general convergence result which applies to a wide range of numerical schemes ('monotone schemes') for nonlinear possibly degenerate elliptic (or parabolic) equation; this type of equation arises naturally in Finance Theory as we will show first. This convergence result was obtained in an article written in collaboration with P.E. Souganidis (1991).

On the other hand, we present several simple numerical schemes for computing the price of different types of 'simple' options: American options, lookback options and Asian options. These schemes are all based on 'splitting methods' and we want to emphasize the fact that this allows also easy extensions for computing the price of more complex options with complicated contracts (cap, floor, ... etc). These schemes also provide examples for which the convergence result of the first part applies. This second part reports on several works in collaboration with J. Burdeau, Ch. Daher & M. Romano (cf. references) which were done in connection with the Research and Development Department of the Caisse Autonome de Refinancement (CDC group).

The article is organized as follows: since the convergence result for numerical schemes relies strongly on the notion of 'viscosity solutions', which is a notion of weak solutions for nonlinear elliptic and parabolic equations, we are first going to present this notion of solutions. In order to introduce it, as a motivation, we examine in the first section several examples of equations arising in Finance Theory, and more particularly in options pricing, and we describe the theoretical difficulties in studying them. The second section presents the notion of viscosity solutions itself: we first introduce the notion of continuous viscosity solutions and then we give the extension to the more complicated framework of discontinuous viscosity solutions which is an unavoidable tool to obtain the general convergence result for numerical schemes which is given in the third section. Then several comments on the assumptions are given and finally, in the fourth section, we present some numerical schemes in option pricing models which are based on splitting methods.

1

2   Examples of Parabolic Equations Arising in Finance Theory

In the classical framework of the theory of Black and Scholes, the stock price $(S_s)_{s \geq t}$ for time $t \geq 0$ is the solution, in the risk-neutral probability, of the stochastic differential equation

$$dS_s = S_s(rds + \sigma dW_s), \quad S_t = S, \tag{2.1}$$

where $(W_s)_{s \geq t}$ is a standard Brownian motion in $\mathbb{R}$. The constants or functions r and σ are known as being respectively the short term interest rate and the so-called volatility. In all the following examples, we will always assume we are in this framework.

We refer to Black & Scholes (1973), Cox & Rubinstein (1985), Cox, Ross & Rubinstein (1979), Duffie (1988) and Ingersoll (1987) for a complete presentation of options pricing theory and its financial background.

1. Classical European Options (Call): Black–Scholes Equation

It is well known that the price of the European call is given for $S \geq 0$ and for $0 \leq t \leq T$ by

$$u(S,t) = \mathbb{E}\left[e^{-r(T-t)}(S_T - K)^+ | \mathcal{F}_t\right]$$

where T is the maturity of the option, K its strike and where $(\mathcal{F}_t)_t$ is the filtration associated to the Brownian motion. The derivation of this type of representation formula for the price of the options is described, for example, in Karatzas & Shreve (1988).

In this case, the function u is a solution of the celebrated Black–Scholes Equation

$$-\frac{\partial u}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} - rS\frac{\partial u}{\partial S} + ru = 0 \quad \text{in } \mathbb{R}^+ \times (0,T),$$

with the terminal data

$$u(S,T) = (S - K)^+ \quad \text{in } \mathbb{R}^+.$$

In this very simple case, there is no problem since u is given by an explicit formula. But it can be interesting to take into account more complicated models with, for example, a non-constant interest rate r and/or a non-constant volatility σ. In order to do this, one needs an adapted theoretical tool to study the equation and efficient numerical schemes to provide accurate approximations of the price u and also of $\partial u/\partial S$ which gives the hedging portfolio.

From a theoretical point of view, there is a difficulty due to the degeneracy of the equation for $S = 0$. To avoid it, a natural idea is to make the change of variable

$$v(x,t) = u(e^x, t) \quad \text{for } x \in \mathbb{R}, \, t \in (0,T),$$

which leads to the equation

$$-\frac{\partial v}{\partial t} - \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial x^2} - \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial v}{\partial x} + rv = 0 \quad \text{in } \mathbb{R} \times (0,T),$$

and then to use on this transformed equation the classical PDE theories (Sobolev spaces, ... etc). We refer the reader to the book of Lamberton & Lapeyre (1992) where this approach is described.

But using such PDE theories leads to work with weighted Sobolev spaces because of the exponential growth of the solutions at infinity and it is never pleasant to have to use these heavy techniques. Moreover, these weighted Sobolev spaces have *a priori* no clear connection with the probabilistic formula of representation for u and their use does not seem to be natural. So it would be convenient to have a theoretical tool to avoid them and to avoid also the exponential change.

2. American Options (Put): Variational Inequalities
In the pricing of American options, because of the possibility of early exercise, the price u is given by a stopping time problem. In the case of a Put, one has

$$u(S,t) = \inf_{\theta \text{ s.t.}} \mathbb{E}\left[e^{-r(\theta-t)}(K - S_\theta)^+ | \mathcal{F}_t\right]$$

where 's.t.' means that θ has to be a stopping time with respect to $(\mathcal{F}_t)_t$.

It is well known that the price u of the option solves the variational inequality

$$\text{Min}\left(-\frac{\partial u}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} - rS\frac{\partial u}{\partial S} + ru, u - (K - S)^+\right) = 0 \quad \text{in } \mathbb{R}^+ \times (0,T),$$

with the terminal data

$$u(S,T) = (K - S)^+ \quad \text{in } \mathbb{R}^+.$$

For the pricing of American options, we refer the reader to Bensoussan (1984) and Karatzas (1988); the more general theory of optimal stopping time control problems is described in Bensoussan & Lions (1978).

The problem is here obviously more complicated: indeed, there is no explicit formula for u – even for constant coefficients r and σ – and we face a nonlinear problem with the same degeneracy as in the case of the European options above.

3. Lookback Options
Lookback options are options on the running maximum of the stock price, a typical example of terminal pay-off being

$$(\max_{0 \le t \le T} S_t - S_T)^+.$$

This problem presents non-Markovian features: indeed, at time $0 < t \leq T$, in order to compute the price of the option, one has to know not only the current stock price S_t but also the value of the running maximum $\max\limits_{0 \leq \tau \leq t} S_\tau$.

Therefore, the price of the option not only depends on t and on S_t but also on the running maximum $\max\limits_{0 \leq \tau \leq t} S_\tau$. To take this fact into account, one has to introduce a new variable Z which carries the past information and the associated process $(Z_s)_s$ given for $s \geq t$ by

$$Z_s = \max \left(Z, \max_{t \leq \tau \leq s} S_\tau \right) ,$$

the idea being that, for $Z = \max\limits_{0 \leq \tau \leq t} S_\tau$, then $Z_s = \max\limits_{0 \leq \tau \leq s} S_\tau$.

To obtain the price of the lookback option, one has to consider the function u, depending on S, Z and t, which is a solution of the problem

$$-\frac{\partial u}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} - rS\frac{\partial u}{\partial S} + ru = 0 \quad \text{in } \{S < Z\} ,$$

$$-\frac{\partial u}{\partial Z} = 0 \quad \text{in } \{S > Z\} ,$$

with the terminal data

$$u(S, Z, T) = (Z - S)^+ \quad \text{in } \mathbb{R}^+ .$$

Indeed, the price of the lookback option at time $0 \leq t \leq T$ and for a current stock price S is given by $u(S, \max\limits_{0 \leq \tau \leq t} S_\tau, t)$.

The main new remark here is that we have to solve a degenerate equation in the domain $\{S < Z\}$ since there are no second-order derivatives with respect to Z in the equation. In fact, it can be shown (cf. Barles, Daher & Romano (1994)) that the above problem reduces to this equation in $\{S < Z\}$ with the oblique derivative boundary condition

$$-\frac{\partial u}{\partial Z} = 0 \quad \text{on } \{S = Z\} .$$

In this simple case, there is an explicit formula for u but again as soon as we consider non-constant coefficients r and σ or if we want to consider some 'American' type features in the option, adapted analytical and numerical tools are needed to study u.

The pricing of lookback options in the case of constant coefficients is studied in Conze (1990). The optimal control problems on the running maximum of a diffusion process are considered in Heinricher & Stockbridge (1991) by probabilistic methods and in Barron (1993) and in Barles, Daher & Romano (1994). In these two last papers, the applications to lookback options are described.

4. Asian Options (Options on the Average)

In this case, the terminal pay-off of the option is typically given by

$$\left(\frac{1}{T} \int_0^T S_\tau d\tau - S_T \right)^+ .$$

The main characteristics of these options are almost the same as for the lookback options; the non-Markovian feature of the problem leads us to introduce the process

$$Z_s = Z + \frac{1}{T} \int_t^s S_\tau d\tau ,$$

for $t \le s$.

To compute the price of the Asian option, one has to consider the solution u of the equation

$$-\frac{\partial u}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} - rS\frac{\partial u}{\partial S} + ru - \frac{S}{T}\frac{\partial u}{\partial Z} = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^+ \times (0, T),$$

with the terminal data

$$u(S, Z, T) = (Z - S)^+ \quad \text{in } \mathbb{R}^+ .$$

The price of the Asian option at time $0 \le t \le T$ and for a current stock price S is then given by $u\left(S, \frac{1}{T}\int_0^t S_\tau d\tau, t\right)$.

Again we face here a degenerate equation, but this time, even with constant coefficients, there is no explicit formula which can be used for a practical point of view.

We refer to Barles, Daher & Romano (1994), Ingersoll (1987) and to Rogers & Shi (1995) for a PDE approach of the pricing of Asian options.

5. Portfolio Management

The last example we want to give is a more complicated example taken from the work of Tourin & Zariphopoulou (1993) where a lot of difficulties are gathered. The equation satisfied by the value-function $v(x, y)$ of the optimal investment-consumption problem is the following

$$\text{Min}\left\{ \inf_{c \ge 0}\left(-\frac{1}{2}\sigma^2 y^2 v_{yy} - (rx - c)v_x - byv_y - U(c) + \beta v\right),\right.$$

$$\left. -v_y + (1 + \lambda)v_x, -(1 - \mu)v_x + v_y \right\} = 0$$

in the domain

$$\{x + (1 - \mu)y \ge 0 \quad \text{and} \quad x + (1 + \lambda)y \ge 0\} ,$$

where the subscripts in the equation mean differentiation with respect to x or y, where $\sigma, r, b, \lambda, \mu$ are constant coefficients, $\beta > 0$ and U is some given utility function.

We do not want to enter too much into the details of this equation and the underlying portfolio management problem but we want to point out the main difficulties one encounters here: we have, at the same time, a *fully nonlinear* and *degenerate* equation with *gradient constraints* and with a *state-constraint boundary condition*. All these difficulties together imply that the theoretical and numerical treatment of this equation is very delicate.

We conclude this section by summarizing the main characteristics of all the above equations: these equations are nonlinear and degenerate. This implies that they have in general no classical solutions ('smooth' solutions). Therefore, a notion of 'weak' solutions is needed in order to make sense of the equations. But as soon as one defines a notion of weak solution, several difficulties occur such as nonuniqueness problems, for example.

On the other (positive) hand, all these equations are *degenerate elliptic equations*, i.e. they can be written as

$$H(x, u, Du, D^2u) = 0 \quad \text{in } \Omega , \qquad (2.2)$$

where Ω is a domain in $\mathbb{R}^N$ and where H is, say, a continuous, real-valued function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}^n$, $\mathcal{S}^n$ being the space of $N \times N$ symmetric matrices, and which satisfies the *ellipticity condition*

$$H(x, u, p, M) \leq H(x, u, p, N) \quad \text{if } M \geq N , \qquad (2.3)$$

for any $x \in \Omega$, $u \in \mathbb{R}$, $p \in \mathbb{R}^N$.

This ellipticity property is a key property for defining the notion of viscosity solutions for the equations (2.2): this fact will become clear in the next section. From now on, we will always assume it is satisfied by the equations we consider.

Remark All the above examples (except the last one) lead to parabolic equations i.e., in particular, to time-dependent equations. To rewrite these equations in the form (2.2), one has to set $x = (y, t)$ where t is the time variable and y is the space variable (typically $y = S$ in examples 1 and 2, $y = (S, Z)$ in examples 3 and 4). In these cases, D and D^2 stand respectively for the gradient and for the matrix of second derivatives with respect to $x = (y, t)$ and not only to the space variables. It is clear that parabolic equations are a particular case of degenerate elliptic equations since these equations contain no second derivatives with respect to t.

3   The Notion of Viscosity Solutions

The notion of viscosity solutions was introduced by Crandall & Lions (1983) (see also Crandall, Evans & Lions (1984)) for solving problems related to

first-order Hamilton–Jacobi Equations. We refer the reader to the 'User's Guide' of Crandall, Ishii & Lions (1992) for a complete presentation of this notion of solutions and to the book of Fleming & Soner (1993) where the applications to deterministic and stochastic optimal control theory are also described.

In order to introduce the notion of viscosity solutions, we give an equivalent definition of the notion of classical solution which uses only the Maximum Principle.

Theorem: (Classical solutions and Maximum Principle) $u \in C^2(\Omega)$ *is a classical solution of*

$$H(x, u, Du, D^2u) = 0 \ \text{ in } \ \Omega \,,$$

where H is a continuous function satisfying (2.3), **if and only if**

$\forall \varphi \in C^2(\Omega)$, *if $x_0 \in \Omega$ is a local maximum point of $u - \varphi$, one has*

$$H(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0 \,,$$

and

$\forall \varphi \in C^2(\Omega)$, *if $x_0 \in \Omega$ is a local minimum point of $u - \varphi$, one has*

$$H(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0 \,.$$

$\square$

The proof of this result is very simple: the first part of the equivalence just comes from the classical properties

$$Du(x_0) = D\varphi(x_0) \,, \quad D^2u(x_0) \leq D^2\varphi(x_0) \,,$$

at a maximum point x_0 of $u - \varphi$ (recall that u and φ are smooth) or

$$Du(x_0) = D\varphi(x_0) \,, \quad D^2u(x_0) \geq D^2\varphi(x_0) \,,$$

at a minimum point x_0 of $u - \varphi$. One has just to use these properties together with the ellipticity property (2.3) of H to obtain the inequalities of the theorem.

The second part is a consequence of the fact that we can take $\varphi = u$ as test-function and therefore $H(x_0, u(x_0), Du(x_0), D^2u(x_0))$ is both positive and negative at any point x_0 of Ω since any $x_0 \in \Omega$ is both a local maximum and minimum point of $u - u$.

Now we simply remark that the equivalent definition of classical solutions which is given here in terms of test-functions φ does not require the existence of first and second derivatives of u. For example, the continuity of u is sufficient to give a meaning to this equivalent definition; so we use this formulation to define viscosity solutions.

Definition: (Continuous Viscosity Solutions) $u \in C(\Omega)$ is a viscosity solution of
$$H(x, u, Du, D^2u) = 0 \quad \text{in } \Omega \,,$$
where H is here a continuous function satisfying (2.3), *if and only if*
$$\forall \varphi \in C^2(\Omega), \text{ if } x_0 \in \Omega \text{ is a local maximum point of } u - \varphi, \text{ one has}$$

$$H(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \le 0 \,,$$

and
$$\forall \varphi \in C^2(\Omega), \text{ if } x_0 \in \Omega \text{ is a local minimum point of } u - \varphi, \text{ one has}$$

$$H(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \ge 0 \,.$$

$\square$

We now give a few concrete examples of equations where there is a unique viscosity solution but no smooth solutions.

The first example is

$$\frac{\partial u}{\partial t} + \left| \frac{\partial u}{\partial x} \right| = 0 \quad \text{in } \mathbb{R} \times (0, +\infty) \,. \tag{3.1}$$

It can be shown that the function u defined in $\mathbb{R} \times (0, +\infty)$ by

$$u(x, t) = -(|x| + t)^2 \,,$$

is the unique viscosity solution of (3.1) in $C(\mathbb{R} \times (0, +\infty))$. It is worth remarking in this example that u is only continuous for $t > 0$ despite the initial data
$$u(x, 0) = -x^2 \quad \text{in } \mathbb{R} \,,$$
is in $C^\infty(\mathbb{R})$. In particular, this problem has no smooth solution as it is generally the case for such nonlinear hyperbolic equations.

Moreover, if we consider (3.1) together with the initial data

$$u(x, 0) = |x| \quad \text{in } \mathbb{R} \,, \tag{3.2}$$

then the functions $u_1(x, t) = |x| - t$ and $u_2(x, t) = (|x| - t)^+$ are two 'generalized' solutions in the sense that they satisfy the equation almost everywhere (at each of their points of differentiability). This problem of nonuniqueness is solved by the notion of viscosity solutions since it can be shown that u_2 is the unique continuous viscosity solution of (3.1)-(3.2). In that case, the notion of viscosity solutions selects the 'good' solution which is in that example the value-function of the associated deterministic control problem (cf. Fleming & Soner (1993)).

For second-order equations, non-smooth solutions appear generally as a consequence of the degeneracy of the equation as in the following example

$$\frac{\partial u}{\partial t} - x^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{in } \mathbb{R} \times (0, +\infty) \tag{3.3}$$

$$u(x,0) = |x|^\alpha \quad \text{in } \mathbb{R}, \tag{3.4}$$

where $0 < \alpha < 1$. The unique uniformly continuous viscosity solution of this problem is

$$u(x,t) = |x|^\alpha e^{\alpha(\alpha-1)t} \quad \text{in } \mathbb{R} \times (0, +\infty),$$

and u is only Hölder continuous in x. The singularity of u at $x = 0$ exists because the equation is degenerate at $x = 0$.

Now we turn to the problem of taking in account the *boundary conditions*: this is a well known difficulty with degenerate equations since losses of boundary data may occur.

We consider for example the Dirichlet problem

$$\begin{cases} H(x, u, Du, D^2u) = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where g is a given continuous function.

In order to solve this Dirichlet problem, a classical idea consists in considering the approximate problem

$$\begin{cases} -\varepsilon \Delta u_\varepsilon + H(x, u_\varepsilon, Du_\varepsilon, D^2u_\varepsilon) = 0 & \text{in } \Omega, \\ u_\varepsilon = g & \text{on } \partial\Omega. \end{cases}$$

Indeed, by adding a $-\varepsilon\Delta$ term, we regularize the equation in the sense that one can expect to have more regular solutions for this approximate problem – typically in $C^2(\Omega) \cap C(\overline{\Omega})$ –.

We assume that this is indeed the case, that this regularized problem has a smooth solution u_ε and, moreover, that $u_\varepsilon \to u$ in $C(\overline{\Omega})$. We forget for the moment that the uniform convergence of u_ε to u on $\overline{\Omega}$ implies that $u = g$ on $\partial\Omega$ and we look for boundary conditions for u.

It is easy to see that the continuous function u satisfies in the viscosity sense

$$\begin{cases} H(x, u, Du, D^2u) = 0 & \text{in } \Omega, \\ \text{Min}(H(x, u, Du, D^2u), u - g) \leq 0 & \text{on } \partial\Omega, \\ \text{Max}(H(x, u, Du, D^2u), u - g) \geq 0 & \text{on } \partial\Omega, \end{cases}$$

where, for example, the 'Min' inequality on $\partial\Omega$ means

$\forall \varphi \in C^2(\overline{\Omega})$, *if $x_0 \in \partial\Omega$ is a maximum point of $u - \varphi$ on $\overline{\Omega}$, one has*

$$\text{Min}(H(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)), u(x_0) - g(x_0)) \leq 0.$$

The proof of the above claim is not difficult: it first consists in showing that *strict* local maximum of minimum points of $u - \varphi$ are limits of local maximum of minimum points of $u_\varepsilon - \varphi$ and then an easy passage to the limit concludes. It remains to remark that the definitions of viscosity solutions obtained by considering 'strict local maximum of minimum points of $u - \varphi$' or 'local maximum of minimum points of $u - \varphi$' are equivalent (cf. Crandall, Ishii & Lions (1992)).

The interpretation of this new problem can be done by setting the equation in $\overline{\Omega}$ instead of Ω. To do so, we introduce the function G defined by

$$G(x, u, p, M) = \begin{cases} H(x, u, p, M) & \text{if } x \in \Omega , \\ u - g & \text{if } x \in \partial\Omega . \end{cases}$$

The above argument shows that the function u is a viscosity solution of

$$G(x, u, Du, D^2u) = 0 \quad \text{on } \overline{\Omega} ,$$

iff

$$G_*(x, u, Du, D^2u) \leq 0 \quad \text{on } \overline{\Omega}$$
$$G^*(x, u, Du, D^2u) \geq 0 \quad \text{on } \overline{\Omega}$$

where G_* and G^* stand respectively for the lower semicontinuous and upper semicontinuous envelopes of G. Indeed, the 'Min' and the 'Max' above are nothing but G_* and G^* on $\partial\Omega$.

In the same way, for general boundary conditions,

$$F(x, u, Du) = 0 \quad \text{on } \partial\Omega ,$$

we introduce the function G defined by

$$G(x, u, p, M) = \begin{cases} H(x, u, p, M) & \text{if } x \in \Omega , \\ F(x, u, p) & \text{if } x \in \partial\Omega . \end{cases}$$

The function u is said to be a viscosity solution of

$$G(x, u, Du, D^2u) = 0 \quad \text{on } \overline{\Omega}$$

if and only if it is a viscosity solution in Ω and if

$$G_* \leq 0 \quad \text{on } \partial\Omega \quad \Longleftrightarrow$$
$$\text{Min}(H(x, u, Du, D^2u), F(x, u, Du)) \leq 0 \text{ on } \partial\Omega$$

and

$$G^* \geq 0 \quad \text{on } \partial\Omega \quad \Longleftrightarrow$$
$$\text{Max}(H(x, u, Du, D^2u), F(x, u, Du)) \geq 0 \text{ on } \partial\Omega .$$

Remark The above example of the Dirichlet problem shows what our convergence result should be able to do: on one hand, it should take into account in a general setting this type of passage to the limit with any kind of boundary conditions (and this is the reason for introducing the formulation of the Gs). On an other hand, it should avoid the uniform convergence property on the u_ε which does not allow boundary layers and losses of boundary data (and this is the reason for introducing discontinuous viscosity solutions now).

Now we give the general definition of discontinuous viscosity solutions.

Definition: (Discontinuous Viscosity Solutions) *A locally bounded upper semicontinuous (usc in short) function u is a viscosity subsolution of the equation*

$$G(x, u, Du, D^2u) = 0 \quad \text{on } \overline{\Omega}$$

if and only if

$\forall \varphi \in C^2(\overline{\Omega})$, *if $x_0 \in \overline{\Omega}$ is a maximum point of $u - \varphi$, one has*

$$G_*(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0 .$$

A locally bounded lower semicontinuous (lsc for short) function v is a viscosity supersolution of the equation

$$G(x, u, Du, D^2u) = 0 \quad \text{on } \overline{\Omega}$$

if and only if

$\forall \varphi \in C^2(\overline{\Omega})$, *if $x_0 \in \overline{\Omega}$ is a minimum point of $u - \varphi$, one has*

$$G^*(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0 .$$

A solution is a function whose usc and lsc envelopes are respectively viscosity sub- and supersolutions of the equation. □

The first reason for introducing such a complicated formulation is to unify the convergence result we present in the next section: we incorporate in the function G the equation together with the boundary condition and this avoids the need of having a different result for each type of boundary condition. The possibility of handling discontinuous sub- and supersolutions is a key point in the convergence proof.

4 Convergence of Numerical Schemes

A numerical scheme approximating the equation

$$G(x, u, Du, D^2u) = 0 \text{ on } \overline{\Omega} , \tag{4.1}$$

is written in the following way

$$S(\rho, x, u^\rho(x), u^\rho) = 0 \text{ on } \overline{\Omega}$$

where S is a real-valued function defined on $\mathbb{R}^+ \times \overline{\Omega} \times \mathbb{R} \times B(\overline{\Omega})$ where $B(\overline{\Omega})$ is the set of bounded functions defined pointwise on $\overline{\Omega}$. We do not denote this space by $L^\infty(\Omega)$ since there is no measure theory involved here and, moreover, we are considering functions defined pointwise and not only almost everywhere.

We assume that the scheme satisfies the following assumptions.

Stability

For any $\rho > 0$, the scheme has a solution u^ρ. Moreover, u^ρ is uniformly bounded, i.e. there exists a constant $C > 0$ s.t.

$$-C \le u^\rho \le C \quad \text{on } \overline{\Omega} ,$$

for any $\rho > 0$.

Consistency

For any smooth function ϕ, one has:

$$\liminf_{\substack{\rho \to 0 \\ y \to x \\ \xi \to 0}} \frac{S(\rho, y, \phi(y) + \xi, \phi + \xi)}{\rho} \ge G_*(x, \phi(x), D\phi(x), D^2\phi(x))$$

and

$$\limsup_{\substack{\rho \to 0 \\ y \to x \\ \xi \to 0}} \frac{S(\rho, y, \phi(y) + \xi, \phi + \xi)}{\rho} \le G^*(x, \phi(x), D\phi(x), D^2\phi(x)).$$

Monotonicity
$$S(\rho, x, t, u) \le S(\rho, x, t, v) \quad \text{if } u \ge v$$
for any $\rho > 0$, $x \in \overline{\Omega}$, $t \in \mathbb{R}$ and $u, v \in B(\overline{\Omega})$.

Strong Comparison Result

If u is a usc subsolution of the equation (4.1) and if v is an lsc supersolution of the equation (4.1), then

$$u \le v \qquad \text{on } \overline{\Omega} .$$

The result is the following.

Theorem *Under the above assumptions, the solution u^ρ of the scheme converges uniformly on each compact subset of $\overline{\Omega}$ to the unique viscosity solution of the equation.* $\square$

Sketch of the proof We just describe the main steps.

1. We set

$$\underline{u}(x) = \liminf_{\substack{\rho \to 0 \\ y \to x}} u^\rho(y) \, ,$$

and

$$\overline{u}(x) = \limsup_{\substack{\rho \to 0 \\ y \to x}} u^\rho(y) \, .$$

The monotonicity and consistency assumptions on S imply that $\overline{u}$ and $\underline{u}$ are respectively sub and supersolutions of the limiting equation.

2. By the Strong Comparison Result for the equation (4.1), we have

$$\overline{u} \leq \underline{u} \quad \text{on } \overline{\Omega} \, .$$

3. But, by definition

$$\underline{u} \leq \overline{u} \quad \text{on } \overline{\Omega} \, .$$

Therefore

$$\overline{u} = \underline{u} \quad \text{on } \overline{\Omega} \, ,$$

and this equality implies the uniform convergence of u^ρ to $u := \overline{u} = \underline{u}$ as a simple variation on the proof of Dini's Theorem. □

Remark The above proof is based on the so-called 'half-relaxed limits method' which was introduced by Perthame and the author (1987). This method allows passages to the limit in fully nonlinear elliptic PDEs with only a uniform bound on the solutions. It is worth mentioning that – because one needs only this uniform bound – this method lets us treat problems where boundary layers occur.

Now we discuss the assumptions on the scheme.

Consistency

For the 'interior points' of Ω where the function G is generally continuous, the consistency requirement is equivalent to

$$\frac{S(\rho, x, \phi(x), \phi)}{\rho} \to G(x, \phi(x), D\phi(x), D^2\phi(x))$$

when $\rho \to 0$ uniformly on compact subsets of Ω, for any smooth function ϕ. We recover here a more standard formulation and the apparent complexity of the consistency assumption above just comes from the fact that we want to handle at the same time the boundary conditions in a general setting which leads to a discontinuous function G.

Strong Comparison Result

This is the key result to get the convergence. Such results exist in the following cases:

  – For first-order equations: optimal Strong Comparison Results have been proved for all kinds of 'classical' equations and boundary conditions (cf. Barles (1994)).

  – For second-order equations: optimal results are available for 'Neumann' type boundary conditions; for 'Dirichlet' type boundary conditions, optimal results exist when the boundary condition is assumed in the classical sense (cf. Crandall, Ishii & Lions (1992) and references therein). For the 'generalized' Dirichlet boundary conditions – where losses of boundary data may occur (which implies that the equation is degenerate) – only the semilinear case is well understood (cf. Barles & Burdeau (1994)). The case of the general Dirichlet problem (fully nonlinear degenerate equation) is still open.

Monotonicity

This assumption can be understood with the following table:

$S(\rho, x, u(x), u) = 0$	$G(x, u, Du, D^2u) = 0$
If $x \in \Omega$ is a maximum point of $u - \phi$, one has	If $x \in \Omega$ is a maximum point of $u - \phi$, one has
Discrete Maximum Principle	**Maximum Principle**
$u \leq \phi + \xi$ where $\xi = u(x) - \phi(x)$.	$Du(x) = D\phi(x)$ $D^2u(x) \leq D^2\phi(x)$
Monotonicity	**Ellipticity**
$S(\rho, x, \phi(x) + \xi, \phi + \xi) \leq \cdots$	$G(x, u, D\phi(x), D^2\phi(x)) \leq \cdots$
$S(\rho, x, u(x), u) = 0$	$G(x, u, Du(x), D^2u(x)) = 0$

It is clear enough from this table that the monotonicity assumption plays, for the numerical scheme, exactly the same role as the ellipticity assumption for the nonlinear PDEs we consider. Therefore

$$\text{Monotonicity} \iff \text{Discrete Ellipticity}$$

First examples

For the sake of simplicity, we first consider classical schemes approximating the heat equation in one dimension

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{in } \mathbb{R} \times (0, T) \,,$$

with a given initial condition

$$u(x, 0) = u_0(x) \quad \text{in } \mathbb{R} \,,$$

where u_0 is, say, a continuous bounded function.

We use below the standard notation in numerical analysis: u_j^n denotes an approximation of $u(n\Delta t, j\Delta x)$ for $n \in \mathbb{N}$ and $j \in \mathbb{Z}$ where Δt and Δx are respectively the mesh size in t and in x. We refer the reader to Glowinski, Lions & Tremolieres (1976) and to Raviart & Thomas (1983) for an introduction to basic methods in numerical analysis.

- **The Standard Implicit Scheme**

If $u^n := (u_j^n)_j$ is known, one can compute $u^{n+1} = (u_j^{n+1})_j$ by solving

$$u_j^{n+1} - u_j^n - \frac{\Delta t}{(\Delta x)^2}(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}) = 0 \,.$$

The above equation has to be read as

$$S\left((n+1)\Delta x, j\Delta t, u_j^{n+1}, u_{j+1}^{n+1}, u_{j-1}^{n+1}, u_j^n\right) = 0 \,;$$

in other words, the above equation is the equation of the scheme at the point $((n+1)\Delta x, j\Delta t)$, the role of the variable ' u^ρ ' is played here by $(u_{j+1}^{n+1}, u_{j-1}^{n+1}, u_j^n)$.

It is clear since we have a '$-$' in front of u_{j+1}^{n+1}, u_{j-1}^{n+1} and u_j^n that this scheme is an unconditionally monotone scheme (there is no condition either on Δt or on Δx). A consequence of this property is also that it is an unconditionally stable scheme since the monotonicity property implies that the scheme satisfies the Maximum Principle i.e.

$$\max_j |u_j^n| \leq \max_j |u_j^0| \,,$$

for any $n \in \mathbb{N}$. And the boundedness of the initial data implies the boundedness of each u^n.

- **The Standard Explicit Scheme**

$$u_j^{n+1} - u_j^n - \frac{\Delta t}{(\Delta x)^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) = 0 \,.$$

We interpret this equality as above; in order to have the monotonicity property, in particular with respect to u_j^n, one should have the classical Courant–Friedrichs–Levy condition

$$\frac{2\Delta t}{(\Delta x)^2} \le 1 \ .$$

If this condition holds, we have a monotone and stable scheme.

Remark Before giving examples of numerical schemes which can be used in options pricing, we want to mention that a general class of schemes for which the above convergence result applies are, in optimal control theory, those which are based on the Dynamic Programming Principle; we refer to Kushner (1977, 1984) for the description of these schemes.

5   Numerical Schemes in Options Pricing: Splitting Methods

We present in this section numerical schemes for computing the price of different types of classical options; these schemes – or more sophisticated schemes based on similar ideas – were implemented at the Caisse Autonome de Refinancement (CDC group). We do not pretend that these schemes are the most efficient in each cases (indeed they are not!). Our aim is to present simple examples to emphasize the advantages of splitting methods.

The reason for using splitting methods was the following: we wanted to build a program for computing the price of a wide variety of options and the use of splitting methods allows us to have a very modular program since the idea is to treat the equations and the constraints separately, one after the other in the right order. In that way, to add more constraints (cap, floor, partially American options, ... etc) is almost costless.

For the sake of simplicity, we are going to present these schemes on simplified equations presenting the same features: we essentially replace below the Black–Scholes equation by the heat equation. Moreover we also present them under the form of approximation schemes; to deduce numerical schemes from them being completely straightforward. It will be clear in each case that the assumptions for the convergence of these schemes are satisfied and we leave the checking to the reader.

1. American options: Variational Inequalities

$$\text{Max}\left(\frac{\partial u}{\partial t} - \Delta u, u - \psi\right) = 0 \quad \text{in } \mathbb{R}^N \times (0, T) \ ,$$

with a given initial condition.

The scheme
1st Step given u^n, we solve

$$\begin{cases} \dfrac{\partial w}{\partial t} - \Delta w = 0 & \text{in } \mathbb{R}^N \times (0, \Delta t), \\[2mm] w(x, 0) = u^n(x) & \text{in } \mathbb{R}^N, \end{cases}$$

and we set

$$u^{n+\frac{1}{2}}(x) = w(x, \Delta t) \quad \text{in } \mathbb{R}^N.$$

2nd Step

$$u^{n+1} = \text{Inf}(u^{n+\frac{1}{2}}, \psi^{n+1}),$$

where $\psi^{n+1}(x) = \psi(x, (n+1)\Delta t)$ for $x \in \mathbb{R}^N$, i.e.

$$u^{n+1} = \text{Inf}(S(\Delta t)u^n, \psi^{n+1}),$$

where S is the semigroup associated with the heat equation.

This scheme is a very classical one: the second step is, in fact, a projection on the convex set $\tilde{K} := \{u \in C(\mathbb{R}^N); \ u \le \psi^{n+1}\}$ and it is generally associated with a Conjugate Gradient Method.

In this example, the claim we made at the beginning of this section becomes more clear: we first treat the equation part by solving the heat equation and then the constraint part by imposing $u \le \psi$. Notice that if this constraint had been put only on some part of the space $\mathbb{R}^N \times (0, T)$, the same type of scheme could have been used. Finally we mention that cap and floor constraints are treated in that way.

2. Lookback Options
The simplified problem is

$$\frac{\partial u}{\partial t} - \Delta_x u = 0 \qquad \text{in } \{|x| < z\},$$

and

$$-\frac{\partial u}{\partial z} = 0 \qquad \text{in } \{|x| > z\},$$

with an initial condition given.

The scheme
1st Step for any fixed z, we solve

$$\begin{cases} \dfrac{\partial w}{\partial t} - \Delta_x w = 0 & \text{in } \mathbb{R}^N \times (0, \Delta t), \\[2mm] w(x, 0) = u^n(x, z) & \text{in } \mathbb{R}^N, \end{cases}$$

and we set

$$u^{n+\frac{1}{2}}(x, z) = w(x, z, \Delta t).$$

2nd Step

$$u^{n+1}(x, z) = \begin{cases} u^{n+\frac{1}{2}}(x, z) & \text{if } |x| < z, \\ \\ u^{n+\frac{1}{2}}(x, |x|) & \text{if } |x| \geq z. \end{cases}$$

This scheme takes into account the different roles of the x and z variables in the equation. It is worth noticing that, in step 1, we solve nondegenerate problems with, for each z, the same equation: in practice, this means that we have to factorize only once the matrix associated with the Δ_x operator. Of course, in concrete computations, step 1 is performed for a finite set of z, namely for the grid points.

Despite these advantages and the fact that we know that this scheme is convergent, it is not very accurate: it is a first-order accurate scheme and the error made for a reasonable number of grid points is not satisfactory. We refer the reader to Barles, Burdeau, Daher & Romano (1995) for an improvement of this scheme by still using splitting methods but in a slightly different way and with a better accuracy.

3. Options on the Average

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - x \frac{\partial u}{\partial z} = 0 \quad \text{in } \mathbb{R} \times \mathbb{R} \times (0, T),$$

with the initial condition given.

The scheme
1st Step for any fixed z, we solve

$$\begin{cases} \dfrac{\partial w}{\partial t} - \dfrac{\partial^2 w}{\partial x^2} = 0 & \text{in } \mathbb{R} \times (0, \Delta t), \\ \\ w(x, 0) = u^n(x, z) & \text{in } \mathbb{R}, \end{cases}$$

and we set
$$u^{n+\frac{1}{2}}(x, z) = w(x, z, \Delta t) \quad \text{in } \mathbb{R}.$$

2nd Step for any fixed x, we solve

$$\begin{cases} \dfrac{\partial w}{\partial t} - x \dfrac{\partial w}{\partial z} = 0 & \text{in } \mathbb{R} \times (0, \Delta t), \\ \\ w(x, 0) = u^{n+\frac{1}{2}}(x, z) & \text{in } \mathbb{R}. \end{cases}$$

and we set
$$u^{n+\frac{1}{2}}(x, z) = w(x, z, \Delta t) \quad \text{in } \mathbb{R}.$$

This scheme is closer to the classical idea of splitting methods, which can be explained in the following way: in order to solve numerically

$$\frac{\partial u}{\partial t} + F_1([u]) + F_2([u]) = 0 \quad \text{in } \mathbb{R}^N \times (0, \Delta t),$$

where $[u]$ stands for (u, Du, D^2u), we apply successively the schemes

$$u^{1/2} = u^0 + \Delta t F_1([u^0]),$$

and

$$u^1 = u^{1/2} + \Delta t F_2([u^{1/2}]).$$

To justify this method, we perform the following formal computations, replacing $u^{1/2}$ in the second equality by its value taken from the first one

$$
\begin{aligned}
u^1 &= u^0 + \Delta t F_1([u^0]) + \Delta t F_2([u^0 + \Delta t F_1([u^0])]) \\
&= u^0 + \Delta t F_1([u^0]) + \Delta t F_2([u^0]) + O\left((\Delta t)^2\right) \\
&\simeq u^0 + \Delta t \left(F_1([u^0]) + F_2([u^0])\right).
\end{aligned}
$$

Therefore within terms of order $O\left((\Delta t)^2\right)$, we have indeed a numerical approximation of the equation.

It is clear, in this example, that the splitting methods allows us to treat differently (using a variety of schemes, for instance) the different parts of an equation or of a complex problem and this is their main advantage.

To conclude this article, we want also to mention that another difficulty in numerically solving equations arising in finance comes from the fact that they are set in unbounded domains. The first step, which consists in approximating these equations by a problem posed in bounded domains, is not obvious since the solutions may not have a well known behavior at infinity (cf. for example the case of lookback options).

It is shown in Barles, Daher & Romano (1995) that the convergence when we let the domain tend to infinity is governed by phenomena of Large Deviations type and therefore completely artificial boundary conditions on the boundary of the domain of computations lead (theoretically) to an exponentially small error inside the domain (only a boundary layer is or should be observed). Nonetheless, this result is theoretical and a good guess on these artificial boundary conditions can really improve the accuracy.

References

Barles G. (1994) *Solutions de viscosité des équations de Hamilton–Jacobi*, Mathématiques et Applications (SMAI), **17**, Springer Verlag.

Barles G. & Burdeau J. (1995) 'The Dirichlet problem for semilinear second-order degenerate elliptic equtions and applications to stochastic exit time control problem', *Comm. in PDE* **20**, 129–178.

Barles G., Burdeau J., Daher Ch. & Romano M. (1995), in preparation.