ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

EDITED BY G.-C. ROTA

Volume 66

Eigenspaces of Graphs

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Eigenspaces of graphs

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CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge CB2 1RP

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, United Kingdom
40 West 20th Street, New York, NY 10011-4211, USA
10 Stamford Road, Oakleigh, Melbourne 3166, Australia

First published 1997

A catalogue record of this book is available from the British Library

Library of Congress cataloguing in publication data

Cvetković, Dragoš M.
Eigenspaces of graphs / Dragoš Cvetković, Peter Rowlinson, Slobodan Simić.
p. cm. – (Encyclopedia of mathematics and its applications; 66)
Includes bibliographical references (p. –) and index.
ISBN 0 521 57352 1 (hc)
1. Graph theory. 2. Spectral theory I. Rowlinson, Peter. II. Simić, Slobodan.
III. Title. IV. Series: Enclyclopedia of mathematics and its applications; v. 66.
QA166.C83 1997
511/.2–dc20 96-2860 CIP

ISBN 0 521 57352 1 hardback

Transferred to digital printing 2004

For Nevenka, Carolyn and Vesna

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Cambridge University Press
0521573521 - Eigenspaces of Graphs
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Preface

The foundations of spectral graph theory were laid in the fifties and sixties, as a result of the work of a considerable number of mathematicians. Most of the early results are, like this book, concerned with the relation between spectral and structural properties of a graph. The investigation of such a relationship was proposed explicitly by Sachs [Sac1] and Hoffman [Hof5], although in effect it had already been initiated in an earlier article by Collatz and Sinogowitz [CoSi]. This seminal paper appeared in 1957, but our bibliography contains two references prior to this date: the unpublished thesis of Wei [Wei] from 1952, and a summary (also unpublished) of a 1956 paper by Lihtenbaum [Lih] communicated at the 3rd Congress of Mathematicians of the U.S.S.R.

Another origin of the theory of graph spectra lies beyond mathematics. In quantum chemistry, an approximative treatment of non-saturated hydrocarbons introduced by E. Hückel [Hüc] yields a graph-theoretical model of the corresponding molecules in which eigenvalues of graphs represent the energy levels of certain electrons. The connection between Hückel's model of 1931 and the mathematical theory of graph spectra was recognized many years later in [GüPr] and [CvGu1], and thereafter exploited extensively by many authors, both chemists and mathematicians.

In his thesis [Cve7], Cvetković identified 83 papers dealing with eigenvalues of graphs which had appeared before 1970. Ten years later, almost all of the results related to the theory of graph spectra published before 1978 were summarized in the monograph *Spectra of Graphs* by Cvetković, Doob and Sachs [CvDS], a book which is almost entirely self-contained; only a little familiarity with graph theory and matrix theory is assumed. Its bibliography contains 564 items, most of which were published between 1960 and 1978. It was supplemented in 1988 by *Recent Results in the Theory of Graph Spectra* by Cvetković, Doob, Gutman and Torgašev [CvDGT]. This reviews the results in spectral graph theory from the period 1978-1984, and provides over 700 further references from the mathematical and chemical literature. There are additional references from areas such as physics, mechanical engineering, geography and the social sciences. Although many papers contain only minor results, and some present rediscoveries of known results, the large number of references indicates the rapid rate of growth of spectral graph theory. The third edition of *Spectra of Graphs*, published in 1995, contains an appendix which describes recent developments in the subject.

This book deals with eigenspaces of graphs, and although one cannot speak about eigenvectors without mentioning eigenvalues, or vice versa, the emphasis is on those parts of spectral theory where the structure of eigenspaces is a dominant feature, thus complementing the 'eigenvalue part of the theory' described in *Spectra of Graphs*. For the most part, the eigenspaces considered are those of a $(0, 1)$-adjacency matrix of a finite undirected graph.

Chapters 1 and 2 review 'old' results on eigenvalues and eigenvectors respectively, while the remaining chapters are devoted to 'new' results and techniques. The eigenspace corresponding to the largest eigenvalue (or *index*) of a connected graph is one-dimensional, and in Chapter 3 a spanning eigenvector is used to identify the graphs with extremal index in various families of graphs. The discussion of graph spectra in the first chapter reveals the limitations of the spectrum as a means of characterizing a graph, and motivates the search for further algebraic invariants such as the graph angles considered in Chapters 4 and 5. Angles also have a role in Chapter 6, where the theory of matrix perturbations is applied to adjacency matrices: one can then describe the behaviour of the index of a graph when it undergoes a local modification such as the addition or deletion of an edge or vertex. Graph angles arise from a geometric approach to eigenspaces that leads in Chapter 7 to the notion of a star partition of vertices, an important concept which enables one to construct 'natural' bases for the eigenspaces of a graph. Implications for the graph isomorphism problem are the subject of current research, and this is described in Chapter 8. Some miscellaneous results are gathered together in Chapter 9, and there are two appendices: one contains some classical results from matrix theory, and the other is a table of graph angles.

The authors are indebted to Mladen Cvetković for assistance with the preparation of a LATEX version of the first draft of the text. The

contents of the second author's article on graph perturbations in *Surveys in Combinatorics 1991* (ed. A.D. Keedwell, Cambridge University Press, 1991) have been included, without significant change, in Chapters 3 and 6. With few other exceptions, the results in Chapters 2 to 9 have not previously appeared in book form.

Finally, the authors gratefully acknowledge individual financial support from the following sources over the past ten years: the British Council, the Carnegie Foundation, the Mathematical Institute of the Serbian Academy of Sciences, the Science and Engineering Research Council, the University of Belgrade and the University of Stirling.

January 1996 D.C., P.R., S.S.

Acknowledgements

The authors are grateful to the following publishers for permission to reproduce, without significant change, sections of the articles cited: The Charles Babbage Research Center [Row7], Gordon and Breach Publishers [RoYu], Elsevier Science Inc. [CvRS1], Elsevier Science Publishers BV [CvRS2], Springer-Verlag GmbH [Row11].