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Dynamical Systems and Ergodic Theory

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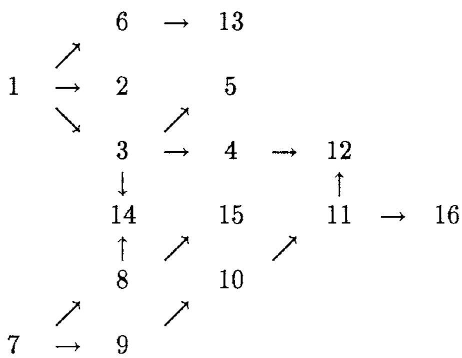
CONTENTS

INTRODUCTION

This book is intended as an introduction to both dynamical systems and ergodic theory. Our aim is to give a direct and detailed introduction to the basic theory, suitable as a text for advanced undergraduate students or beginning graduate students in mathematics.

The notes divide naturally into three parts. The first part (chapters 1-6) concentrates on topological dynamics. The second part (chapters 7-12) deals with ergodic theory and measurable dynamics. The third part (chapters 13-16) consists of more advanced material to supplement the two earlier parts.

Each of the first two parts is intended to be essentially self-contained, as is illustrated by the following diagram of the relationships between chapters:



The areas of dynamical systems and ergodic theory are rich in connections with other subjects (e.g. number theory, geometry, statistics, mathematical physics, biology, etc.). In the course of these notes we have tried to motivate the general theory with some important applications (particularly to number theory, in chapter 2 and chapter 16).

There are already a number of excellent books on dynamical systems and ergodic theory (e.g. Devaney's *Introduction to Chaotic Dynamical Systems*) Walters' *Introduction to Ergodic Theory* and *Introduction to the Modern Theory of Dynamical Systems* by Katok and Hasselblatt).

To pre-empt any comparison with these fine texts, we should emphasize that this book is intended to be a more modest introduction to the subject. The reader who would like to find out more about dynamical systems and ergodic theory will find much more in these books.

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PRELIMINARIES

1. Conventions. The book is divided into 16 chapters, each subdivided into sections numbered in order (e.g. chapter 12, section 3 is numbered **12.3**).

Within each chapter results (Theorems, Propositions or Lemmas) are labelled by the chapter and then the order of occurrence (e.g. the fifth result in chapter 3 is `Proposition 3.5`). The exceptions to this rule are: sublemmas which are presented within the context of the proof of a more important result (e.g. the proof of Theorem 2.2 contains Sublemmas 2.2.1 and 2.2.2); and corollaries (the corollary to Theorem 5.5 is Corollary 5.5.1).

We denote the end of a proof by ■.

Finally, equations are numbered by the chapter and their order of occurrence (e.g. the fourth equation in chapter 5 is labelled (5.4))

2. Notation. We shall use the standard notation: $\mathbb{R}$ to denote the *real numbers*; $\mathbb{Q}$ to denote the *rational numbers*; $\mathbb{Z}$ to denote the *integer numbers*; $\mathbb{N}$ to denote the *natural numbers*; and $\mathbb{Z}^+$ to denote the non-negative integers. We use the convenient convention that: $\mathbb{R}/\mathbb{Z} = \{x + \mathbb{Z} : x \in \mathbb{R}\}$ (which is homeomorphic to the standard unit circle); $\mathbb{R}^2/\mathbb{Z}^2 = \{(x_1, x_2) + \mathbb{Z}^2 : (x_1, x_2) \in \mathbb{R}^2\}$ (which is homeomorphic to the standard 2-torus); etc. However, for $x \in \mathbb{R}$ we denote the corresponding element in $\mathbb{R}/\mathbb{Z}$ by $x \pmod 1$ (and similarly for $\mathbb{R}^2/\mathbb{Z}^2$, etc.).

We denote the interior of a subset A of a metric space by $\mathrm{int}(A)$, and we denote its closure by $\mathrm{cl}(A)$.

If $T : X \to X$ denotes a continuous map on a compact metric space then T^n $(n \geq 1)$ denotes the composition with itself n times.

If $T : I \to I$ is a C^1 map on the unit interval $I = [0, 1]$ then T' denotes its derivative.

3. Prerequisites in point set topology (chapters 1-6). The first six chapters consist of various results in topological dynamics for which the only prerequisite is a working knowledge of point set topology for metric spaces. For example:

THEOREM A (BAIRE). *Let X be a compact metric space; then if $\{U_n\}_{n\in\mathbb{N}}$ is a countable family of open dense sets then $\bigcap_{n\in\mathbb{N}} U_n \subset X$ is dense.*

THEOREM B (SEQUENTIAL COMPACTNESS). *Let X be a metric space; then X is compact if and only if every sequence $(x_n)_{n\in\mathbb{N}}$ in X contains a convergent subsequence.*

THEOREM C (ZORN'S LEMMA). *Let Z be a set with a partial ordering. If every totally ordered chain has a lower bound in Z then there is a minimal element in Z.*

Two good references for this material are [4] and [5]

4. Pre-requisites in measure theory (chapters 7-12). Chapters 7-12 form an introduction to ergodic theory, and suppose some familiarity (if not expertise) with abstract measure theory and harmonic analysis. The following results will be required.

THEOREM D (RIESZ REPRESENTATION). *There is a bijection between*

(1) *probability measures μ on a compact metric space X (with the Borel sigma algebra),*
(2) *Continuous linear functionals $c : C^0(X) \to \mathbb{R}$,*

given by $c(f) = \int f d\mu$.

THEOREM E. *Let $(X, \mathcal{B}, \mu)$ be a measure space. For every linear functional $\alpha : L^1(X, \mathcal{B}, \mu) \to L^1(X, \mathcal{B}, \mu)$ there exists $k \in L^\infty(X, \mathcal{B}, \mu)$ such that $\alpha(f) = \int f \cdot k d\mu$, $\forall f \in L^1(X, \mathcal{B}, \mu)$* [3, p.121].

In proving invariance of measures in examples the following basic result will sometimes be assumed.

THEOREM F (KOLMOGOROV EXTENSION). *Let $\mathcal{B}$ be the Borel sigma-algebra for a compact metric space X. If μ_1 and μ_2 are two measures for the Borel sigma-algebra which agree on the open sets of X then $m_1 = m_2$* [3, p. 310].

The following terminology will be used in the chapter on ergodic measures. Given two probability measures μ, ν we say that μ is *absolutely continuous* with respect to ν if for every set $B \in \mathcal{B}$ for which $\nu(B) = 0$ we have that $\mu(B) = 0$. We write $\mu << \nu$ and then we have the following result.

THEOREM G (RADON-NIKODYM). *If μ is absolutely continuous with respect to μ then there exists a (unique) function $f \in L^1(X, \mathcal{B}, d\nu)$ such that for any $A \in \mathcal{B}$ we can write $\mu(A) = \int_A f d\nu$.*

We usually write $f = \frac{d\mu}{d\nu}$ and call this the *Radon-Nikodym derivative* of μ with respect to ν.

We call two measures μ, ν *mutually singular* if there exists a set $B \in \mathcal{B}$ such that $\mu(A) = 0$ and $\nu(A) = 1$. We then write $\mu \perp \nu$.

In chapter 8 we shall need a passing reference to Lebesgue spaces. A *Lebesgue space* is a measure space which is measurably equivalent to the

the union of unit intervals (with the usual Lebesgue measure) with at most countably many points (with non-zero measure).

In chapter 11 we shall use the following result.

THEOREM H (DOMINATED CONVERGENCE). *Let* $h \in L^1(X, \mathcal{B}, \mu)$ *and let* $(f_n)_{n\in\mathbb{Z}^+}$
$\subset L^1(X, \mathcal{B}, \mu)$*, with* $|f_n(x)| \le h(x)$*, converge (almost everywhere) to* $f(x)$*; then* $\int f_n d\mu \to \int f d\mu$ *as* $n \to +\infty$*.*

Good general references for this material are [1], [2], [3].

5. Subadditive sequences. A simple result which proves its worth several times in these notes is the following.

THEOREM F (SUBADDITIVE SEQUENCES). *Let* $(a_n)_{n\in\mathbb{N}}$ *be a sequence of real numbers such that* $a_{n+m} \le a_n + a_m$*,* $\forall n, m \in \mathbb{N}$ *(i.e. a subadditive sequence); then* $a_n \to a$*, as* $n \to +\infty$*, where* $a = \inf\{a_n \colon n \ge 1\}$

PROOF. First note that $a_n \le a_1 + a_{n-1} \le \ldots \le na_1$, and so $a \le a_1$ For $\epsilon > 0$ we choose $N > 0$ with $a_N < N(a+\epsilon)$. For any $n \ge 1$ we can write $n = kN + r$, where $k \ge 0$ and $1 \le r \le N-1$. Then

$$a_n \le a_{kN} + a_r \le ka_N + a_r \le ka_N + \sup_{1\le r\le N} a_r$$

and we see that

$$\limsup_{n\to+\infty} \frac{a_n}{n} \le \limsup_{k\to+\infty} \frac{ka_N + \sup_{1\le r\le N} a_r}{kN} = \frac{a_N}{N} \le a + \epsilon.$$

This shows that $\frac{a_n}{n} \to a$, as required.

■

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