CHAPTER 1

EXAMPLES AND BASIC PROPERTIES

In this chapter we shall introduce some of the basic dynamical properties associated to continuous maps $T : X \to X$ on compact metric spaces.

1.1 Examples

To set the stage, we begin with some standard examples of continuous maps (transformations) which will be used to illustrate different properties.

EXAMPLE 1 (DOUBLING MAP). Let X denote the unit interval with its endpoints identified, $X = \mathbb{R}/\mathbb{Z}$. Define a continuous map $T : X \to X$ by $T(x) = 2x \pmod 1$, i.e.

$$Tx = \begin{cases} 2x & \text{if } 0 \leq x < \frac{1}{2}, \\ 2x - 1 & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

This is usually called the "doubling map", since it doubles distances on X. An equivalent formulation would be if we let $X = K := \{z \in \mathbb{C} : |z| = 1\}$ and then define $T : X \to X$ by $T(e^{2\pi i\theta}) = e^{2\pi i 2\theta}$, where 0 ¡ $\theta < 1$. This is equivalent in the sense that there is a homeomorphism $\rho : \mathbb{R}/\mathbb{Z} \to K$ given by $\rho(\theta + \mathbb{Z}) = e^{2\pi i\theta}$ which relates the two transformations. We shall return to this notion of equivalence (or conjugacy) in chapter 3.

EXAMPLE 2 (ROTATIONS ON THE CIRCLE). Let $X = \mathbb{R}/\mathbb{Z}$ and fix a number $\alpha \in [0, 1)$. We define a homeomorphism $T : X \to X$ by $T(x) = x + \alpha \pmod 1$, i.e.

$$Tx = \begin{cases} x + \alpha \text{ if } 0 \leq x + \alpha \leq 1, \\ x + \alpha - 1 \text{ if } x + \alpha > 1. \end{cases}$$

(An equivalent formulation would be if we let $X = K$ and then define $T : X \to X$ by $T(e^{2\pi i\theta}) = e^{2\pi i(\theta+\alpha)}$. This is equivalent in the sense that the homeomorphism $\rho : \mathbb{R}/\mathbb{Z} \to K$ given by $\rho(t) = e^{2\pi it}$ relates the two transformations.)

EXAMPLE 3 (SHIFT MAP). For $k \geq 2$ let $X_k = \prod_{n\in\mathbb{Z}}\{1, 2, \ldots, k\}$ denote the space of all sequences taking values $\{1, 2, \ldots, k\}$ indexed by $\mathbb{Z}$. In order to define a metric we first associate to two sequences $x = (x_n)_{n\in\mathbb{Z}}$ and

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$x = (x_n)_{n\in\mathbb{Z}}$ an integer $N(x,y) = \min\{N \geq 1 : x_N \neq y_N \text{ or } x_N \neq y_N\}$. We define a metric on X_k by

$$d(x,y) = \begin{cases} \left(\frac{1}{2}\right)^{N(x,y)} & \text{if } x \neq y, \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 1.1. *X_k is a compact space.*

PROOF. We shall actually show that X_k is sequentially compact. Let $x^{(k)} = (x_n^{(k)})_{n\in\mathbb{Z}}$ $(k = 1, 2, 3, \ldots)$ be a sequence in X_k; then we need to show that there exist a point $x \in X_k$ and a sub-sequence $x^{(k_l)} \to x$ $(l = 1, 2, 3, \ldots)$.

First observe that the zeroth terms $x_0^{(k)}$ $(k = 1, 2, 3 \ldots)$ must take some value in $\{1, 2, \ldots, k\}$ infinitely often. Choose such an $x_0 \in \{1, 2, \ldots, k\}$ with $x_0^{(k)} = x_0$, for infinitely many m. We continue inductively: For $l > 0$, choose $x_l \in \{1, 2, \ldots, k\}$ and $x_{-l} \in \{1, 2, \ldots, k\}$ such that $x_{-l}^{(m)} = x_{-l}, \ldots, x_0^{(m)} = x_0, \ldots, x_l^{(m)} = x_l$, say, for infinitely many m. Finally, we define $x = (x_l)_{l\in\mathbb{Z}}$. For each $l \geq 0$ we choose $m_l := m$ such that $x_{-l}^{(m)} = x_{-l}, \ldots, x_0^{(m)} = x_0, \ldots, x_l^{(m)} = x_l$; then $d(x^{(m_l)}, x) \leq \frac{1}{2^l}$ and so $d(x^{(m_l)}, x) \to 0$ as $l \to +\infty$. ■

DEFINITION. We can define a map $\sigma : X_k \to X_k$ by $(\sigma x)_n = x_{n+1}$, $\forall n \in \mathbb{Z}$, i.e.

$$\sigma : (\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots) \longmapsto (\ldots, x_{-1}, x_0, x_1, x_2, x_3, \ldots).$$

Since this map shifts sequences by one place it is called the *shift map*.

LEMMA 1.2. *The map $\sigma : X_k \to X_k$ is a homeomorphism.*

PROOF. To show continuity we observe that if $x \neq y$ and $d(x,y) = \left(\frac{1}{2}\right)^N$ then we know that $x_i = y_i$ for $-N \leq i \leq N$. Thus we have that $(\sigma x)_i = x_{i+1} = y_{i+1} = (\sigma y)_i$ for $i = -(N+1), \ldots, N-1$. This means that $d(\sigma x, \sigma y) \leq \left(\frac{1}{2}\right)^{N-1} = \frac{1}{2} d(x,y)$ and we see that σ is continuous.

Clearly $\sigma : X_k \to X_k$ is invertible (since the inverse transformation $\sigma^{-1} : X_k \to X_k$ simply shifts sequences back one place). Finally, the inverse map $\sigma^{-1} : X_k \to X_k$ is continuous by the same sort of argument as above. ■

1.2 Transitivity

In this section we shall introduce some basic properties of continuous maps $T : X \to X$ on compact metric spaces X.

DEFINITION. We say that a homeomorphism $T : X \to X$ of a compact metric space X is *transitive* if there exists a point $x \in X$ such that its orbit

$\{T^n x : n \in \mathbb{Z}\} = \{\ldots, T^{-2}x, T^{-1}x, x, Tx, T^2x, \ldots,\}$ is dense in X. We call such a point $x \in X$ a *transitive point.*

We say that a continuous map $T : X \to X$ of a compact metric space X is *(forward) transitive* if there exists a point $x \in X$ such that its *orbit* $\{T^n x : n \in \mathbb{Z}^+\} = \{x, Tx, T^2x, \ldots,\}$ is dense in X. We call such a point $x \in X$ a *(forward) transitive point.*

We can check each of the examples in section 1.1 for this property.

EXAMPLE 1. We shall show that this example is forward transitive when $k = 2$, other cases ebing similar. Consider the sequence $1, 2, 11, 12, 21, 22, 111, 112, 121, 122, 221, \ldots, 222, 1111, \ldots$. We can write down $x_n \in \{1, 2\}$, $n \geq 0$, as the nth term in the sequence

$$121112212221111121211222 21 \ldots 2221111 \ldots .$$

Finally, consider the point $x \in [0, 1]$ given by the series $x = \sum_{n=0}^{+\infty} \frac{(x_n - 1)}{2^{n+1}}$. We claim that the point x is a (forward) transitive point. Observe that

$$Tx = 2\left(\sum_{n=0}^{+\infty} \frac{(x_n - 1)}{2^{n+1}}\right) \pmod 1 = x_0 + \sum_{n=0}^{+\infty} \frac{(x_{n+1} - 1)}{2^{n+1}} \pmod 1$$
$$= \sum_{n=0}^{+\infty} \frac{(x_{n+1} - 1)}{2^{n+1}}.$$

Similarly, $T^k x = \sum_{n=0}^{+\infty} \frac{x_{n+k}}{2^n}$.

To show that the set $\{T^n x : n \geq 0\}$ is dense it suffices to show that for each interval of the form $\left[\frac{p}{2^l}, \frac{p+1}{2^l}\right]$, with $0 \leq p \leq 2^l - 1$, we can find $N \geq 0$ with $T^N x \in \left[\frac{p}{2^l}, \frac{p+1}{2^l}\right]$. Given p we can write it in binary form as $i_0 \ldots i_{n-1}$, with $i_0, \ldots, i_{n-1} \in \{0, 1\}$. But for some N we can find $x_N = i_0, x_{N+1} = i_1, \ldots, x_{N+n-1} = i_{n-1}$. This means that $T^N x \in \left[\frac{p}{2^l}, \frac{p+1}{2^l}\right]$, as required.

EXAMPLE 2. There are two different cases, depending on whether or not α is irrational.

First assume that α is irrational, then the map $T : X \to X$ can be shown to be transitive (and even forward transitive) where $x = 0$, say. It suffices to show that the orbit $\{T^n 0\}_{n \in \mathbb{Z}^+}$ is dense. Since this is an infinite set in $\mathbb{R}/\mathbb{Z}$ we can choose $x \in \mathbb{R}/\mathbb{Z}$ and a sub-sequence $n_i \to +\infty$ with $T^{n_i} 0 = n_i \alpha \pmod 1 \to x$. For any sufficiently small $\epsilon > 0$ we can choose $n_i > n_j$ with $|T^{n_i} 0 - x| < \frac{\epsilon}{2}$ and $|T^{n_j} 0 - x| < \frac{\epsilon}{2}$ and thus $|T^{n_i} 0 - T^{n_j} 0| = |T^{n_i - n_j} 0| < \epsilon$. Moreover, $T^{n_i} 0 \neq T^{n_j} 0$, since if not this would contradict α being irrational. Thus the points $T^{(n_i - n_j)k} 0$, $k \geq 1$, form an ϵ-dense subset of $\mathbb{R}/\mathbb{Z}$. Since ϵ can be chosen arbitarily small this completes the proof of transitivity.

Now assume that $\alpha = \frac{p}{q}$ with $p, q \in \mathbb{Z}$ having no common divisors and $q \neq 0$. For any $x \in X$ the orbit $\{T^n x : n \in \mathbb{Z}\}$ would be a finite set $\{x, x + \frac{1}{q}, \ldots, x + \frac{q-1}{q} \pmod 1\}$. In particular, T is *not* transitive.

EXAMPLE 3. We shall show that this example is forward transitive. The sequence

$$\{x_n\}_{n\in\mathbb{N}} = \{1, 2, \ldots, k, 1, 1, \ldots, 1, k, 2, 1, \ldots, 2, k, \ldots, \underbrace{z_0, z_1, \ldots, z_{N-1}}_{\text{All strings appear}} \ldots\}$$

in X_k (in which all finite strings appear once) is a forward transitive point. To see this choose any point $z \in X_k$ and for any $\epsilon > 0$ choose $N > 0$ sufficiently large that $\left(\frac{1}{2}\right)^N < \epsilon$. If we choose r such that $x_r = z_0, \ldots, x_{r+N-1} = z_{N-1}$ then we see that $(\sigma^r x)_0 = x_r = z_0, \ldots, (\sigma^r x)_{N-1} = x_{r+N-1} = z_{r+N-1}$ and so $d(\sigma^r x, z) \le \left(\frac{1}{2}\right)^N < \epsilon$.

1.3 Other characterizations of transitivity

The following result gives equivalent conditions for a homeomorphism of a compact metric space to be transitive.

THEOREM 1.3. *The following are equivalent.*

(i) $T : X \to X$ *is transitive.*
(ii) *If* U *is an open set with* $TU = U$ *then either* U *is dense or* $U = \emptyset$.
(iii) *If* U, V *are non-empty open sets then for some* $n \in \mathbb{Z}$ *we have that* $T^n U \cap V \neq \emptyset$.
(iv) *The set* $\{x \in X : \text{ the orbit } \{T^n x\}_{n\in\mathbb{Z}} \text{ is dense in } X\}$ *is a dense* G_δ *set (i.e. the intersection of a countable collection of open dense sets).*

PROOF. (i) $\implies$ (ii). Assume $x \in X$ has a dense orbit. Assume that $TU = U \neq \emptyset$. We can choose $n \in \mathbb{Z}$ such that $T^n x \in U$. Moreover, for any $m \in \mathbb{Z}$ we have that $T^m x \in T^{m-n} U = U$. Since the orbit of x is dense (i.e. $\cup_{m\in\mathbb{Z}} T^m x \subset X$ is dense) we see that U is dense.
(ii) $\implies$ (iii). The T-invariant union $\cup_{n\in\mathbb{Z}} T^n U$ is dense in X by assumption (ii). Thus $\cup_{n\in\mathbb{Z}} T^n U \cap V \neq \emptyset$ and so $\exists n \in \mathbb{Z}$ with $T^n U \cap V \neq \emptyset$.
(iii) $\implies$ (iv). Consider a dense set $\{x_n\}_{n\in\mathbb{N}}$ and consider the balls of radius $\frac{1}{k}, k \ge 1$, denoted by $B\left(x_n, \frac{1}{k}\right)$. We can identify

$$\{x \in X : \{T^m x\}_{m\in\mathbb{Z}} \text{ is dense in } X\} = \cap_{n=0}^{+\infty} \cap_{k=1}^{+\infty} \cup_{m=-\infty}^{+\infty} T^m B\left(x_n, \frac{1}{k}\right)$$

(i.e. $\forall n \ge 0, \forall k \ge 1, \exists m \in \mathbb{Z}$ with $T^m x \in B\left(x_n, \frac{1}{k}\right)$).
(iv) $\implies$ (i). This is immediate.

■

REMARK. There is a similar result giving equivalent conditions for forward transitivity [4, p. 128]

1.4 Transitivity for subshifts of finite type

In section 1.1 we defined the shift transformation $\sigma : X_k \to X_k$ on $X_k = \prod_{n\in\mathbb{Z}}\{1, \dots, k\}$. For any closed σ-invariant subset $X \subset X_k$ (i.e. $\sigma(X) = X$) we consider the restriction $\sigma|_X$. We can use the same notation $\sigma : X \to X$.

DEFINITION. Let A be a $k \times k$ matrix with entries 0 or 1. We call the matrix *irreducible* if $\forall 1 \le i, j, \le k$, $\exists N > 0$ such that $A^N(i, j) > 0$.

EXAMPLES. When $k = 3$ the matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ is irreducible. However, the matrix $A' = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is not irreducible. (These properties are readily checked).

DEFINITION. Given a $k \times k$ matrix A with entries 0 or 1 we define

$$X_A = \{(x_n)_{n\in\mathbb{Z}} \in \prod_{n=-\infty}^{\infty} \{1, \dots, k\} : A(x_n, x_{n+1}) = 1, n \in \mathbb{Z}\}.$$

We define the *subshift of finite type* $\sigma : X_A \to X_A$ to be the restriction $\sigma|X_A$.

The following gives necessary and sufficient conditions for $\sigma : X_A \to X_A$ to be transitive.

THEOREM 1.4. *A subshift of finite type $\sigma : X_A \to X_A$ is transitive if and only if A is irreducible.*

PROOF. Assume that σ is transitive. Consider the sets

$$[i]_0 := \{(x_n)_{n\in\mathbb{Z}} \in X_A : x_0 = i\}$$

for $i = 1, \dots, k$. These sets are open. Given $1 \le i, j \le k$ we know that there exists $N > 0$ such that $\sigma^{-N}[j]_0 \cap [i]_0 \ne \emptyset$. Choose $(x_n)_{n\in\mathbb{Z}} \in \sigma^{-N}[j]_0 \cap [i]_0$; then we know that $x_0 = i$ and $x_N = j$. Notice that

$$A^N(i, j) = \sum_{r_1=1}^{k} \dots \sum_{r_{N-1}=1}^{k} A(i, r_1)A(r_1, r_2) \dots A(r_{N-2}, r_{N-1})A(r_{N-1}, j).$$

But since $A(i, x_1) = A(x_1, x_2) = \dots = A(x_{N-1}, j) = 1$ we see that $A^N(i, j) \ge 1$.

Conversely, assume that for $1 \le i, j \le k$ we have that $A^N(i, j) \ge 1$. Given $U, V \ne \emptyset$ open sets we can choose $(i_n)_{n\in\mathbb{Z}} \in U$ and $(j_n)_{n\in\mathbb{Z}} \in V$ such that for $M > 0$ sufficiently large

$$U \supset [i_{-M}, i_{-M-1}, \dots, i_M]_{-M}^{M} := \{(x_n)_{n\in\mathbb{Z}} \in X_A : x_k = i_k, -M \le k \le M\},$$

$$V \supset [j_{-M}, j_{-M-1}, \dots, j_M]_{-M}^{M} := \{(x_n)_{n\in\mathbb{Z}} \in X_A : x_k = j_k, -M \le k \le M\}.$$

By hypothesis we can find $N > 0$ such that $A^N(i_M, j_{-M}) \geq 1$. This means that we can find a string $x'_1, \ldots, x'_{N-1}$ such that $A(i_M, x'_1) = A(x'_1, x'_2) = \ldots = A(x'_{N-1}, j_{-M}) = 1$ and then define

$$x_n = \begin{cases} i_n \text{ if } n \leq M, \\ x'_{n-M} \text{ if } M+1 \leq n \leq M+N-1, \\ j_{n-(2M+N)} \text{ if } M+N \leq n; \end{cases}$$

then we have that $x \in U \cap \sigma^N V$ i.e. $U \cap \sigma^N V \neq \emptyset$.

■

1.5 Minimality and the Birkhoff recurrence theorem

In this section we want to present a simple but important recurrence result, called the *Birkhoff recurrence theorem.* Our starting point is to define the following property.

DEFINITION. A homeomorphism $T : X \to X$ is *minimal* if for every $x \in X$ the orbit $\{T^n x : n \in \mathbb{Z}\}$ is dense in X.

The following is obvious from the definitions

PROPOSITION 1.5. *A minimal homeomorphism is necessarily transitive.*

We can now consider each of the examples from section 1.1 and ask which of these are minimal. Since Example 1 is not a homeomorphism we begin with Example 2.

EXAMPLE 2.

LEMMA 1.6. *When α is irrational then $T(x) = x + \alpha$ is minimal.*

PROOF. It suffices to show that for every $x \in \mathbb{R}/\mathbb{Z}$ and every neighbourhood $(y - \epsilon, y + \epsilon)$ $(y \in \mathbb{R}/\mathbb{Z}, \epsilon > 0)$ we can find $n \geq 1$ such that $T^n(x) \in (y - \epsilon, y + \epsilon)$.

We already know that T is transitive (i.e. there exists at least one transitive point $x_0 \in \mathbb{R}/\mathbb{Z}$ with dense orbit). Fix $y \in \mathbb{R}/\mathbb{Z}$ and use the transitivity to choose a sub-sequence n_i with $T^{n_i} x_0 \to (y - x + x_0)$ as $i \to +\infty$. Thus

$$\begin{aligned} T^{n_i} x &= n_i \alpha + x \pmod 1 \\ &= n_i \alpha + x_0 + (x - x_0) \pmod 1 \\ &= T^{n_i}(x_0) + (x - x_0) \pmod 1 \\ &\to y + (x_0 - x) + (x - x_0) = y \pmod 1. \end{aligned}$$

■

EXAMPLE 3. The shift map is not minimal since it contains a fixed point (e.g. $x = (\ldots, 1, 1, 1, \ldots)$).

The following theorem gives equivalent definitions.

THEOREM 1.7. *Let $T : X \to X$ be a homeomorphism of a compact metric space. The following properties are equivalent.*

(i) *T is minimal.*
(ii) *If $TE = E$ is a closed T-invariant set, then either $E = \emptyset$ or $E = X$.*
(iii) *If $U \neq \emptyset$ is an open set then $X = \cup_{n \in \mathbb{Z}} T^n U$.*

PROOF. *(i)* $\Longrightarrow$ *(ii)* Assume that $TE = E \neq \emptyset$ and choose $x \in E$. Hypothesis (i) gives that $X = \mathrm{cl}\,(\{T^n x\}_{n \in \mathbb{Z}}) \subset E \subset X$.

(ii) $\Longrightarrow$ *(iii)* Given a non-empty open set U let $E = X - (\cup_{n \in \mathbb{Z}} T^n U)$. By construction $TE = E$ and $E \neq X$ (since $U \neq \emptyset$) and so by hypothesis (ii) we have that $E = \emptyset$. Thus $X = \cup_{n \in \mathbb{Z}} T^n U$.

(iii) $\Longrightarrow$ *(i)* Fix $x \in X$ and an open neighbourhood $U \subset X$. Since $x \in T^n U$ for some $n \in \mathbb{Z}$ (by hypothesis (iii)) we have that $T^{-n} x \in U$. This shows that the orbit $\{T^n x\}_{n \in \mathbb{Z}}$ is dense in X.

■

Using property (ii) we get the following suprising result that every homeomorphism contains a minimum homeomorphism.

THEOREM 1.8. *Let $T : X \to X$ be a homeomorphism of a compact metric space X. There exists a non-empty closed set $Y \subset X$ with $TY = Y$ and $T : Y \to Y$ is minimal.*

PROOF. This follows from an application of Zorn's Lemma. Let $\mathcal{E} = \{Z \subset X : TZ = Z\}$ denote the family of all T-invariant subsets of X with the partial ordering by inclusion, i.e. $Z_1 \leq Z_2$ iff $Z_1 \subset Z_2$.

Every totally ordered subset (or "chain") $\{Z_\alpha\}$ has a least element $Z = \cap_\alpha Z_\alpha$ (which is non-empty by compactness of X). Thus by Zorn's lemma there exists a minimal element $Y \subset X$ (i.e. $Y \in \mathcal{E}$ and $Y' \in \mathcal{E}$ with $Y' \leq Y$ implies that $Y = Y'$). By property (ii) of Theorem 1.7 this can be reinterpreted as saying thet $T : Y \to Y$ is minimal.

■

As a corollary we get the following simple but elegant result.

COROLLARY 1.8.1(BIRKHOFF RECURRENCE THEOREM). *Let $T : X \to X$ be a homeomorpism of a compact metric space X. We can find $x \in X$ such that $T^{n_i} x \to x$ for a sub-sequence of the integers $n_i \to +\infty$.*

PROOF. By Theorem 1.8 we can choose a T-invariant subset $Y \subset X$ such that $T : Y \to Y$ is minimal. For any $x \in Y \subset X$ we have the required property.

■

EXAMPLE 2. Consider the case $X = \mathbb{R}/\mathbb{Z}$ and $T : X \to X$ defined by $Tx = x + \alpha \pmod 1$, where α is an irrational number.

Let $\epsilon > 0$; then we can find $n > 0$ (by Birkhoff's theorem) such that $|\alpha n \pmod 1| \leq \epsilon$, i.e. there exists $p \in \mathbb{N}$ such that $-\epsilon \leq \alpha n - p \leq \epsilon$. Rewriting this, we have that for any irrational α, $\exists p, n \in \mathbb{N}$ such that $|\alpha - \frac{p}{n}| \leq \frac{\epsilon}{n}$. This is a (marginal) improvement on the most obvious estimate.

1.6 Commuting homeomorphisms

Let $T_1, \ldots, T_N : X \to X$ be commuting homeomorphisms on a compact metric space X, i.e. $T_i T_j = T_j T_i$ for $1 \leq i, j \leq N$. In this section we shall briefly consider how some of the ideas from the previous section might be modified for such families of maps.

EXAMPLE 4. Consider the simple example of two rotations on the torus $X = \mathbb{R}^n / \mathbb{Z}^n$ of the form

$$T_1(x_1, \ldots, x_n) = (x_1 + a_1^{(1)}, \ldots, x_n + a_n^{(1)}) \pmod 1,$$

$$\vdots$$

$$T_N(x_1, \ldots, x_n) = (x_1 + a_1^{(N)}, \ldots, x_n + a_n^{(N)}) \pmod 1,$$

where $(a_1^{(1)}, \ldots, a_n^{(1)}), \ldots, (a_1^{(N)}, \ldots, a_n^{(N)}) \in \mathbb{R}^n$.

We can consider all closed *simultaneously invariant* sets $A \subset X$, i.e. $T_i A = A$, $i = 1, \ldots, N$. By a similar argument to that before, we can consider the partial order by inclusion on all such closed sets and by applying Zorn's lemma (just as in the proof of Theorem 1.8) we can deduce that there exists a closed set $X_0 \subset X$ such that

(i) $T_i X_0 = X_0$, $i = 1, \ldots, N$.
(ii) whenever $A \subset X_0$ with A closed and $T_i A = A$ for $i = 1, \ldots, N$ then necessarily $A = X_0$.

The following lemma will prove useful in chapter 2.

LEMMA 1.9. *For each open set $U \subset X_0$ we can choose a finite M and $n_{ij} \in \mathbb{Z}$ with $1 \leq i \leq N, 1 \leq j \leq M$ with $X_0 = \cup_{j=1}^{M} (T_1^{n_{1j}} \circ \ldots \circ T_N^{n_{Nj}}) U$.*

PROOF. Clearly $X_0 = \cup_{n_1 \in \mathbb{Z}} \ldots \cup_{n_N \in \mathbb{Z}} (T_1^{n_1} \circ \ldots \circ T_N^{n_N}) U$ (since otherwise the difference $X_0 - (\cup_{n_1 \in \mathbb{Z}} \ldots \cup_{n_N \in \mathbb{Z}} (T_1^{n_1} \circ \ldots \circ T_N^{n_N}) U)$ is a closed (non-empty) set invariant under $T_1, \ldots, T_N$, contradicting property (ii) above). Now by compactness we can choose a *finite* subcover. This completes the proof.

■

To formulate a generalisation of the Birkhoff recurrence theorem to a family of commuting maps is a more substantial exercise, and will be a principal part of chapter 2.

1.7 Comments and references

A wealth of interesting examples can be found in the literature (cf. [1], [2], [3], [4], [5]).

The simple Birkhoff recurrence theorem has a version for commuting homeomorphisms (the multiple Birkhoff recurrence theorem) which we shall describe in Chapter 2. The corresponding result to the Birkhoff recurrence theorem in ergodic theory is the Poincaré recurrence theorem, which appears in section 9.2.

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CHAPTER 2

AN APPLICATION OF RECURRENCE TO ARITHMETIC PROGRESSIONS

In this chapter we shall describe a particularly nice application of the recurrence ideas from chapter 1 to a result in number theory.

2.1 Van der Waerden's theorem

We begin with a simple idea from number theory.

DEFINITION. An *arithmetic progression* is a sequence of integers $\{a + jb\}_{j=0}^{N-1}$ for $a, b \in \mathbb{Z}$ $(b \neq 0)$, $N \geq 1$. We call N the *length* of the arithmetic progression.

EXAMPLES.

(1) The sequence $10, 13, 16, 19, 22$ is an arithmetic progression with $a = 10, b = 3, N = 5$.
(2) The sequence $-4, 0, 4, 8$ is an arithmetic progression with $a = -4, b = 4, N = 4$.

Consider a *partition* of the integers $\mathbb{Z} = B_1 \cup \ldots \cup B_l$ where

(i) $B_i \neq \emptyset$,
(ii) $B_i \cap B_j = \emptyset$ for $i \neq j$.

The main result we want to prove is the following.

THEOREM 2.1 (VAN DER WAERDEN). *Consider a finite partition $\mathbb{Z} = B_1 \cup \ldots \cup B_k$. At least one element B_r in the partition will contain arithmetic progressions of arbitrary length (i.e. $\exists 1 \leq r \leq k$, $\forall N > 0$, $\exists a, b \in \mathbb{Z}$ $(b \neq 0)$ such that $a + jb \in B_r$ for $j = 0, \ldots, N-1$).*

Since an arithmetic progression of length N contains arithmetic progressions of all shorter lengths, this is equivalent to: $\exists N_i \to +\infty$, $\exists a_i, b_i \in \mathbb{Z}$ such that $a_i + jb_i \in B_r$ for $j = 0, \ldots, N_i - 1$.

We give below some simple examples.

EXAMPLES.

(1) If the sets $B_2, \ldots, B_k$, say, in the partition are finite then it is easy to see that B_1 is the element with arithmetic progressions of arbitrary length.