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Contact and Symplectic Geometry

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CONTACT AND SYMPLECTIC GEOMETRY

edited by

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Preface

Between July and December 1994 the Isaac Newton Institute in Cambridge organised a number of activities around the theme "Symplectic Geometry". The present volume contains contributions by some of the participants in the first two workshops. These were devoted respectively to contact and symplectic geometry in dimensions 3 and 4, and to invariants of symplectic manifolds (notably Floer homology). Most of the papers submitted are expository, and it is to be hoped that they will be of use to anyone wanting to learn about a subject which both has very old mathematical roots and which of late has experienced an extraordinary flowering.

I wish to thank everyone at the Isaac Newton Institute, who worked so hard during the six months, and also the other members of the organising committee, Simon Donaldson, Dusa McDuff and Dietmar Salamon, each of whom made a distinct contribution to the success of the meeting. Finally I am grateful to my old friend David Tranah of Cambridge University Press, for his assistance at all stages of publication, and for his patience with my unfamiliarity with electronic book lore.

Charles Thomas

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Introduction

C.B. Thomas

Basic Ideas

An even-dimensional manifold M^{2n} is said to be *symplectic* of it supports a closed 2-form ω of maximal rank, i.e. $d\omega = 0$ and ω^n defines a volume form everywhere. The basic example is provided by the phase space of a Hamiltonian system, otherwise put, by the total space $M^{2n} = T^*N^n$ of the cotangent bundle of any smooth manifold. Other examples are complex Kähler manifolds, i.e. such that the associated almost complex structure J is an isometry over each point, and invariant under parallelism.

In odd dimensions a contact structure is defined by a non-integrable codimension one distribution $\xi \subset TM^{2n+1}$. At least for orientable manifolds, and up to multiplication by a nowhere zero function $f \in C^\infty(M, \mathbb{R})$, this is equivalent to the existence of a 1-form α such that $\alpha_\wedge(d\alpha)^n \neq 0$. The basic example is provided by a constant energy level in the system above, e.g. the unit sphere bundle $P^{2n-1} \subset T^*N^n$. Other examples can be constructed as S^1-fibrations over projective algebraic varieties. Indeed such fibrations are the model for the *contactification* of (M^{2n}, ω); in the reverse direction one can always embed (P^{2n-1}, α) in a (usually non-compact) symplectification. As we shall see, contact manifolds are much easier to construct than symplectic, and as a result subobjects play a particularly important role in the latter category. We say that $Q^{2r} \subseteq M^{2n}$ is symplectic if $\omega|Q$ is non-degenerate and Lagrangian if $\omega|Q = 0$. (The analogue of a Lagrangian submanifold is contact geometry is said to be Legendrian.)

It will be clear from the various sections of this introduction that much symplectic and contact geometry is rooted in complex analysis and complex algebraic geometry. For the necessary background in these subjects the reader can consult [G-H] and [Kr]. See also the very recent short survey [Ja]. Also no introduction would be complete without mentioning Gromov's book [Gr2] -however, like the work of Martin Heidegger, reading this can be compared to walking along a forest path, occasionally emerging into a beautiful clearing.

1 Symplectic 4-manifolds

Let V^4 be a simply-connected 4-dimensional manifold carrying a symplectic form ω. The most familiar examples arise as projective algebraic varieties (Kähler) embedded in some space $\mathbb{C}P^N$, and the existence of ω implies that the structural group of the tangent bundle TV reduces from $SO(4)$ to $U(2)$. Hence we have inclusions

$$\text{Kähler manifolds} \hookrightarrow \text{symplectic manifolds} \hookrightarrow \text{almost complex manifolds},$$

both of which are proper. Even in the simply-connected case one can construct non-Kähler symplectic manifolds using a codimension two connected sum construction, see [Go]. At the other extreme there exist non-integrable almost complex structures on connected sums (in the usual sense) of copies of $\mathbb{C}P^2$ and $\overline{\mathbb{C}P}^2$. That these cannot be symplectic is a consequence of C. Taubes' work on the Seiberg-Witten equations [Tb]. The nature of the forgetful maps is also interesting and is bound up with several conjectures, starting with the question as to whether the map

$$\left\{ \begin{array}{c} \text{1-connected complex} \\ \text{surfaces/deformation} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{1-connected oriented} \\ C^\infty\text{-manifolds/Diff}^+ \end{array} \right\}$$

is injective. This has been shown to be finite to one [F-M], Thm 5.2 and §2.7, and (1-1) for surfaces of elliptic type. In the symplectic category it is known, see the survey of McDuff in this collection, that, if V^4 admits a symplectically embedded copy of $\mathbb{C}P^1$ with positive normal bundle, then V^4 is a rational or ruled complex surface. Morever, again using the Seiberg-Witten equations, C. Taubes has shown that the symplectic structure on $\mathbb{C}P^2$ is unique. Taken together, these examples illustrate the tantalising link between Donaldson and Seiberg-Witten theory. On the one hand these theories are somehow "dual" to one another; on the other Donaldson's symplectic embedding theorem (surveyed below) guarantees the existence of a symplectically embedded surface in $(\mathbb{C}P^2, \omega)$ (with ω = arbitrary), whose properties constrain ω in the way explained by McDuff. The forgetful maps are also illustrated by variants of the Thom conjecture on the representability of homology classes.

Theorem 1.1 [K-M] *If V^4 is a complex surface and $0 \neq \xi \in H_2(V, Z)$ can be represented by a smooth, irreducible holomorphic curve, then this curve has minimal genus among all smooth embeddings.*

What happens when we replace "complex" by "symplectic"? It is known that the analogous question has negative answer if the so-called *symplectic irreductibility conjecture* fails. In its weaker form this states that:

$$\textit{if } V \cong V_1 \# V_2, \text{ then } b_2^+(V_i) = 0 \text{ for } i = 1 \text{ or } 2,$$

and has been shown by Donaldson to hold for Kähler manifolds. In this direction one also has the *symplectic completeness conjecture*:

every smooth, oriented, 1-connected M^4 is diffeomorphic to a connected sum of symplectic manifolds (with both orientations).

Note that this does not say that M^4 is itself symplectic, since forms ω_1, and ω_2 behave badly under the usual connected sum. This is in sharp contrast to the 3-dimensional theory discussed below.

Although no pattern has yet emerged to explain the various examples, symplectic theory in dimension 4 seems to retain much of the flavour of Kähler manifolds. A good illustration of this is given by Taubes' use of the Seiberg-Witten equations, in which he models his argument on the special case of Kähler manifolds, but has to introduce an error term involving the Nijenhuis tensor, which is later deformed away.

2 Contact 3-manifolds

At the beginning of this introduction we observed that in a classical Hamiltonian system the 'constant energy levels' carry a natural contact form. In dimension 3 this observation leads to the definition of a *fillable* contact structure, a class which inherits some of the 'hardness' which we have seen that symplectic 4-manifolds share with complex surfaces. What makes this particularly important is the 'softness' of contact structures in general. In dimension 3, and at least for highly connected manifolds in other dimensions, the obstructions to the existence and classification of contact forms are homotopy theoretic. Put another way, it is relatively easy to construct contact manifolds by surgery, see [We] and [Gr2]. For example, because S^{2n+1} inherits a contact structure from the connection on the S^1-fibration over $\mathbb{C}P^n$, whose curvature is the pull-back of the natural symplectic form, the contact-connected sum of M_1 and M_2 makes sense. As we have seen, to find a symplectic analogue of this construction we must pass to codimension two at least, [Go].

Fillable structures are contained in the possibly wider class of *tight* structures - see the survey based on Y. Eliashberg's lectures in this collection for definitions. Where tight structures exist, they seem to be very important for the geometry of the manifold. For example, the standard structure on S^3 described above is tight, and any other tight structure is isotopic to it. Using this deep result one can show that any diffeomorphism of S^3 extends to a diffeomorphism of D^4, and hence recover very elegantly J. Cerf's theorem that $\Gamma_4 = \{0\}$, [C]. Since Cerf's theorem is the beginning of one proof of the homotopy equivalence of $SO(4)$ and $Diff^+(S^3)$ (Smale's conjecture /Hatcher's theorem) it is not unreasonable to expect contact geometry to throw more light on this area. For example, in the case of the lens space $L(p, q)$ we expect the group $Diff^+$ to be homotopic to the group of oriented isometries. If we could prove this rigorously by contact geometry or otherwise, we would be able to complete the classification of free actions by finite *solvable* groups on S^3, an important step in W. Thurston's geometrisation programme, see J. Rubinstein's survey [R]. Note in this direction that, if the free action is defined by a representation in the group of *strict* contact automorphisms of S^3, we can already show that it is conjugate to a linear action.

Besides describing the basic results in 3-dimensional contact geometry Eliashberg's lectures are also concerned with the notion of a *confoliation*, i.e. with a tangent plane field on an oriented manifold associated with a 1-form α satisfying $\alpha_\wedge d\alpha \geqslant 0$. At the time, i.e. July 1994, it was only possible to sketch the proof of the main theorem stating that a smooth codimension one foliation can be C^0-perturbed to a contact structure. Although the details have since been worked out by Eliashberg and Thurston, in outline the argument stands as originally given. One of its important consequences is that it shows that a *taut* foliation (i.e. one without Reeb components) perturbs to a *tight* contact structure. Since D. Gabai [Ga] has shown that, if F is an oriented surface of minimal genus representing $0 \neq \xi \in H_2(M, \mathbb{Z})$ with M irreducible, then F is a leaf in a taut foliation, the class of 3-manifolds admitting tight contact structures is large.

The Reeb stability theorem shows that there is a certain rigidity attached to codimension one foliations. Speaking very loosely one might say that they set the pattern for contact structures in a way that recalls the relation between Kähler and symplectic structures in dimension four. Another link is provided by the work of H. Geiges and J. Gonzalo, who consider 3-manifolds admitting circles of contact forms $\lambda_1\omega_1 + \lambda_2\omega_2$, $(\lambda_1, \lambda_2) \in S^1$,. Symplectification of this form gives an almost complex structure J on $M^3 \times \mathbb{R}$, which is integrable if and only if the volume forms $\omega_{\lambda_1,\lambda_2} \wedge d\omega_{\lambda_1,\lambda_2}$ are the same for all pairs (λ_1, λ_2). (Such special classes of contact circles can only exist on a restricted class of Seifert fibrations, see the actual survey below.)

3 Higher dimensional implications

The methods and results discussed in the two previous sections have implications for the next few dimensions. Even though these are not covered in the present volume, since they are likely to figure in the next stage of research, it seems worthwhile to say a little about them. Contact structures on 1-connected 5-manifolds are well-understood, see [Ge], the obstructions to both existence and classification being largely topological in nature. Similar results hold for 2-connected 7-manifolds, which are also of interest from the point of view of three quaternionically related contact forms, and, at least in outline, it is clear how to prove similar results in the case when $H_2(M^7, \mathbb{Z}) \neq 0$. This satisfactory situation depends on the 'softness' of contact structures in general, and the obvious intermediate problem of 1-connected symplectic 6-manifolds is wide open. One does at least have a good starting point-namely the topological classification in terms of algebraic invariants due to C.T.C. Wall and others. (Wall restricted attention to spin manifolds with torsion free homology, see [Wa] and [Ž]). Recent developments in the theory of 3-folds provide numerous projective algebraic examples, and the fact that S^6 is at least almost complex allows us to take connected sums of 'almost symplectic' manifolds. Potentially more fruitful is S. Donaldson's symplectic submanifold theorem, which will provide 4-dimensional submanifolds in any putative (M^6, ω), possibly leading to non-existence results.

Another way in which dimension four influences adjacent dimensions is in the field of structure preserving diffeomorphism. In her survey article below. see also [Gr1], McDuff outlines Gromov's description of the homotopy type of $Diff_\omega(\mathbb{C}P^2)$ and $Diff_{\omega_1,\omega_2}(\mathbb{C}P^1 \times \mathbb{C}P^1)$ (split forms). For $\mathbb{C}P^2$ we now know that the symplectic structure is unique, and as a consequence there is only one 'regular' contact structure α on S^5. Furthermore it is a result from the thesis of A. Banyaga [B] that the group of *strict* contact automorphisms is a central extension of

$$[Diff_\omega(\mathbb{C}P^2),\ Diff_\omega(\mathbb{C}P^2)] \text{ by } S^1,$$

i.e. again of the homotopy type of a compact Lie group. These results all suggest that, in the presence of a contact or symplectic form, the nice homotopy theoretic properties of diffeomorphism groups in low dimensions extend to dimensions in which the general methods of surgery also apply. This again opens the way to a profitable interplay between geometry and algebraic topology.

More generally still one is led to speculate whether $Diff_\omega(M^{2n})$ may not be a more tractable group to study than unrestricted orientation preserving diffeomorphisms. In the same way that $Sp(2n, \mathbb{R})$ is a generalisation of $SL(2, \mathbb{R})$ may it not also be true that $Diff_\omega(\mathbb{C}P^n)$ is a more useful generalisation of $Diff^+(S^2)$ than $Diff^+(S^{2n})$?

4 Homology theories

One of the most interesting recent developments has been the construction of various homology theories specially adapted to the study of manifolds equipped with a geometric structure. The basic idea goes back to E. Witten's reformulation of classical Morse theory [Wi], see also the article by D. Salamon [Sa]. The common framework consists in taking a map $f : M \to \mathbb{R}$ with non-degenerate critical points, i.e. at least in the finite dimensional case, such that the Hessian matrix of second partial derivatives is non-singular at each critical point. Defining the *index* to be the number of negative eigenvalues, and c_k to be the number of critical points of index k, one can establish the

Morse inequalities

$$c_k - c_{k-1} + \qquad \pm c_0 \geqslant b_k - b_{k-1} + \cdots \pm b_0.$$

Furthermore homology groups can be calculated from the chain complex

$$\{C_k = \bigoplus_{\substack{x = \text{signed} \\ \text{critical point}}} \mathbb{Z}_{(x)} : k \geqslant 0\}$$

with boundary homomorphisms defined by using the submanifold $W^s(x) \cap W^u(y)$ to count trajectories passing from x to y. Here $W^s(x)$ (respectively $W^u(y)$) stands for the stable manifold at x (respectively unstable manifold at y). This bald description shows that there are severe technical problems to be overcome when we replace a compact manifold M by say its space of contractible loops $\Omega^\circ(M)$ as in Schwarz' survey below. These are overcome in all the variants of *Floer* homology by making a very special choice for the Morse function f. Schwarz considers the variant of this theory, which was originally developed to solve (partially) V.I. Arnold's conjecture that the minimum number of fixed points of an exact symplectomorphism on M is bounded by the sum of the Betti numbers - compare the inequality above. This depends crucially on the fact that $H_k^F(M,\cdot) \cong H_k^{sing}(M,\cdot)$ as additive groups. Dualising to cohomology one can define at least two products, referred to respectively as the *quantum* and *pair of pants* products, which are distinct from the usual singular cup product. (An interesting example, referred to repeatedly during the INI meeting, is given by the moduli space of flat connections on a surface of genus g. The reader may well like to consider it at length for her/himself.) These product structures are the subject of the article by S. Pianikhin, D. Salomon and M. Schwarz, while H.V. Le and K. Ono reinterpret results on the Arnol'd conjecture in terms of cup length in the cohomology ring $H^*(M, \mathbb{F}_2)$. Another variant of Floer (co)homology is explained in the survey by Y. G. Oh. This is a powerful tool in the study of the symplectic topology of *Lagrangian* submanifolds, and among other things Oh discusses applications to Gromov's non-exactness theorem, a lower bound for the symplectic disjunction energy, non-degeneracy of the Hofer metric on the space of Lagrangian submanifolds in T^*M, and Maslov class obstructions to embedding in $\mathbb{C}^N$. Y. Chekanov's contribution is also in this area.

Symplectic geometry has its origin in the study of Hamiltonian dynamical systems, and although these were not the prime object of study in any of the two workshops, the present volume includes a few contributions in this direction, most obviously those of Y. Karshon and V. Ginzburg.

Besides being an attempt to introduce the reader to the wide variety of topics covered in this volume, one of the aims of this introduction has been to suggest various lines for future research. Inevitably this has shown the personal bias of one trained as a differential topologist, but I hope, nonetheless, that it conveys some of the importance and interest of a flourishing subject.

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