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J-curves and the classification of rational and ruled symplectic 4-manifolds

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In this paper, we present a complete classification of rational and ruled symplectic 4-manifolds up to symplectomorphism as well as describing ways of recognizing these manifolds. The proofs are essentially self-contained.

1 Introduction

The classification of symplectic structures on $\mathbf{CP}^2$ has followed a quite simple story: Gromov showed in [2] that the existence of an embedded sympectic 2-sphere in the homology class $[\mathbf{CP}^1]$ of a line implies that the symplectic structure is diffeomorphic to the standard Kähler structure on $\mathbf{CP}^2$, and Taubes showed in [38] that such a sphere always exists if the space is diffeomorphic to $\mathbf{CP}^2$. Both proofs are due to spectacular advances in the application of elliptic PDE methods to symplectic 4-manifolds.

The classification of rational ruled symplectic 4-manifolds, that is the classification of symplectic S^2-fibrations over a 2-sphere, was established by Gromov [2] in a rather special case, and was extended to all cases by McDuff [14]. Her proof was based on two new ideas: the realization that in 4-dimensions cusp-curves do not affect the behaviour of the evaluation map for generic J, and the construction of symplectic sections of a rational ruled 4-manifold with a method suggested by Eliashberg.

However, the classification of symplectic structures on spaces diffeomorphic to S^2-bundles over Riemann surfaces of genus > 0 is more complicated. There have been several attempts to solve this problem: partial results (including a complete solution of the problem for ruled surfaces over the torus) were obtained by McDuff in [14, 21, 23] and by Lalonde in [5]. Actually the main difficulty, as we will see below, is not to derive the existence of an embedded symplectic 2-sphere. It is to classify the symplectic S^2-fibrations on a given S^2-bundle over a Riemann surface of genus > 0. This required more work, both on the *symplectic* and *elliptic* aspects of the problem. The complete classification finally appeared in our recent note [7]. It is based on a simpler and more general geometric argument which reduces the classification of the irrational ruled manifolds to the rational ones, up to the computation of some Gromov invariant on the given irrational ruled manifold. The first step of this reduction – the "cutting and pasting" argument – appeared in McDuff [14], and the second step of the reduction – the transformation of a symplectic deformation into a genuine isotopy via a one-parameter family of symplectic submanifolds – was worked out by Lalonde in [5]. The computation of the relevant Gromov invariant is based on the equivalence of the Gromov and Seiberg-Witten invariants, and is easily reduced to the calculation of a wall-crossing number at the reducible solutions of the Seiberg-Witten equations.

The second theme of this article is the characterization of rational and ruled symplectic manifolds. While developing the theory of J-holomorphic spheres, McDuff discovered that any symplectic 4-manifold which contains a symplectically embedded 2-sphere C of nonnegative self-intersection is the blow-up of a rational or ruled manifold. Taubes's work now applies to show that this remains true if we just assume that C is smoothly embedded. Another beautiful new result was recently proved by Liu [10]. He showed that Gompf's conjecture holds: namely, any minimal symplectic 4-manifold whose canonical class K is such that $K^2 < 0$ is irrational ruled. As a corollary, it follows that any minimal symplectic 4-manifold which admits a metric of positive scalar curvature is rational or ruled: see Liu [10] and Ohta–Ono [31].

2 Statement of the classification theorems

We assume that the reader is familiar with blowing-up and blowing-down in the symplectic category: see for instance [27] or [5], §II, for a convenient summary.

Recall that there are only two S^2-bundles over any given Riemann surface B, the trivial bundle $\pi : B \times S^2 \to B$ and the nontrivial bundle $\pi : M_B \to B$. By analogy with the complex case, such a bundle is said to be a **ruled symplectic manifold** if the total space is equipped with a symplectic form ω which is nondegenerate on each fiber. In this case, we also say that ω is **compatible** with the given ruling π. Similarly, we say that a symplectic 4-manifold (M, ω) is **rational** if it can be obtained from the standard $\mathbf{CP}^2$ by a sequence of blow-ups and blow-downs. (This is slightly unfortunate terminology, since in other contexts the word rational is used to denote a rationality condition on some cohomology class, for instance on the class $[\omega]$ or on some Maslov class.)

An **exceptional sphere** in a symplectic 4-manifold is a symplectically embedded 2-sphere of self-intersection -1.[1] It was shown in [17, 14] that these spheres behave very much like exceptional curves in complex surfaces. In particular, they can be blown down and replaced by a ball of appropriate size. (In fact, if the *weight* $\omega(E)$ of the exceptional curve E is πr^2 the ball

[1] To say that a 2-dimensional submanifold S of M is symplectically embedded is equivalent to saying that the restriction $\omega|_S$ never vanishes.

should have radius r.) If we define a **minimal** symplectic 4-manifold to be one which contains no exceptional spheres, one can reduce every symplectic 4-manifold to a minimal one by blowing down a maximal collection of disjoint exceptional spheres. Moreover, just as in the complex case, this minimal reduction is unique unless it it is rational or ruled: see [18].

It is also convenient to consider manifolds M which are such that the complement $M - C$ of a symplectic 2-submanifold C contains no exceptional spheres. (In this situation (M, C) is sometimes called a **minimal pair**.) Observe that given any such C in M one can obtain a minimal pair by blowing down a maximal collection of disjoint exceptional spheres in $M - C$.

Here are the main results.

Theorem 2.1 (Gromov–McDuff) *Let (M, ω) be a closed symplectic 4-manifold which contains an embedded symplectic 2-sphere C of self-intersection ≥ 0. Then (M, ω) is symplectomorphic to a blow-up either of $\mathbf{CP}^2$ with its standard Kahler structure (which is unique up to scaling by a constant) or of a ruled symplectic manifold. In particular, if $M - C$ is minimal then (M, ω) is either symplectomorphic to $\mathbf{CP}^2$ or is ruled.*

Remarks. (1) In the above theorem, the natural hypothesis would be to assume M minimal. But then one does not recover the case of the topologically non-trivial S^2-bundle over S^2, which coincides with the blow-up of $\mathbf{CP}^2$ at one point. In order to include this case, we must assume only the minimality of $M - C$.

(2) The symplectomorphism of the theorem can be chosen so that the curve C corresponds either to a line or quadric in $\mathbf{CP}^2$ or to a fiber of the S^2-bundle, or to a section of the bundle when the base is the sphere.

(3) This theorem implies that the rational manifolds are actually blow-ups of the standard $\mathbf{CP}^2$ or of the ruled $S^2 \times S^2$ (which by Theorem 2.3 below must also be standard, ie a product). In other words one does not need to take a sequence of blow-ups and blow-downs to arrive at an arbitrary rational symplectic manifold.

Theorem 2.2 *(i)* **(Taubes)** *If (M, ω) is a symplectic manifold and M is smoothly diffeomorphic to $\mathbf{CP}^2$, then ω is diffeomorphic to a standard Kahler form.*

(ii) **(Li–Liu)** *Let $\pi : M \to \Sigma$ be an smooth S^2-bundle over a closed orientable surface. Then any symplectic form ω on M is diffeomorphic to a form which is compatible with the given ruling. Moreover, we can assume that this diffeomorphism acts trivially on homology.*

Theorem 2.3 (McDuff) *Let (M, ω) be a ruled symplectic manifold over a 2-sphere. Then ω is isotopic to a standard Kahler form on M.*

Theorem 2.4 (Lalonde–McDuff) *Let (M,ω) be a ruled symplectic manifold over a closed orientable surface. Then ω is isotopic to a standard Kahler form on M.*

In the last two theorems, the statement is more precisely: any two cohomologous symplectic forms compatible with the ruling of a given bundle $S^2 \hookrightarrow M \to \Sigma$ are isotopic (and therefore isotopic to the standard Kahler form in the given cohomology class).

Remarks. (1) It follows from the last three theorems that symplectic forms on smooth S^2-bundles are determined up to diffeomorphism by the classes $[\omega] \in H^2(M, \mathbf{R})$.

(2) Taubes's work also leads to some general recognition theorems for rational and ruled manifolds. One of the nicest of these is due to Liu, who showed in [10] that Gompf's conjecture holds. Namely every minimal symplectic 4-manifold (M,ω) with $K^2 < 0$ is a ruled surface over a base of genus > 1. Here $K = -c_1(\omega)$ is the canonical class. It follows that any symplectic 4-manifold which admits a symplectic submanifold C with $c_1(C) > 0$ is a blow-up of a rational or ruled manifold, which generalizes a well-known theorem in complex geometry.

As we said above, the classification of symplectic structures on spaces diffeomorphic to $\mathbf{CP}^2$ and the classification of a restricted kind of symplectic structure on $S^2 \times S^2$ was proved by Gromov [2]. Theorem 2.1 is McDuff's classification theorem from [14], which was proved by further developing the pseudoholomorphic techniques introduced in Gromov's seminal work [2]. The proof has two steps. One first shows that when $M - C$ is minimal M contains a symplectically embedded sphere of self-intersection either 0 or 1, and then analyses the diffeomorphism type of M in these two cases.

The second theorem is one of the consequences of Taubes's recent beautiful work on the equivalence of the Gromov and the Seiberg–Witten invariants. He proved part (i) of the theorem in [38] by finding a symplectic sphere of self-intersection 1. Part (ii) was proved by Li and Liu [8, 9] using a direct extension of Taubes's argument. We sketch the outlines of these proofs in §6.3 below.

The third theorem was proved by McDuff in [14]. Its proof is based on an extension of Gromov's techniques and on some new ideas that are discussed further below.

The proof of the fourth theorem involves an extended foray into symplectic geometry. Recall that two symplectic forms are said to be *deformation equivalent* if they can be joined by a path of not necessarily cohomologous symplectic forms, while they are *isotopic* if the path ω_t consists of cohomologous forms. In the latter case, Moser stability implies that there is a path of diffeomorphisms ϕ_t of M starting at the identity such that $\phi_t^*(\omega_t) = \omega_0$. Deformation equivalent forms, on the other hand, while they have the same Gromov invariants may not be geometrically equivalent. More precisely, there are examples of cohomologous and deformation equivalent symplectic forms which are not even diffeomorphic (see [13]), though no such examples are known in dimansion 4.

The first step in the proof of Theorem 2.4 is a cutting and pasting argument which shows that all symplectic forms on a ruled manifold compatible with a given ruling are deformation equivalent. This was proved in McDuff's paper [14] by enlarging the base to make enough room in which to cut open the ruled surface over a set of loops in the base in such a way that the monodromies over each such loop are trivial. This yields a ruled surface over a cell in $\mathbf{R}^2$ with a symplectic form which is standard near the boundary of the cell. One can then complete this to a ruled surface over S^2 and invoke uniqueness for ruled surfaces over S^2. The second step consists in showing that, if ω, ω' are two cohomologous and deformation equivalent symplectic forms compatible with a given ruling, then they are isotopic. A rather complicated geometric way of doing this was proposed in [14] and used in [21, 23] to prove the theorem in special cases, for example when the base is a torus. This procedure was greatly simplified by Lalonde in [5], Lemma I.5. His idea is to transform the one-parameter family of symplectic forms $\omega_{t\in[0,1]}$ by adding at each time an appropriate multiple of the Thom class of a ω_t-symplectic surface Z_t of M and a multiple of the Thom class of a ω_t-symplectic S^2-fiber. This will change the family ω_t into a family ω_t' of constant cohomology class provided that the homology class of the surfaces Z_t is correctly chosen. (This choice depends on the ratios of the slopes of the vectors $[\omega_t]$ and $[\omega_t']$ in $H^2(M, \mathbf{R}) = \mathbf{R}^2$.) This reduces the proof of the fourth classification theorem to the existence of the symplectic surfaces Z_t, which in turn is established by the computation of a Gromov invariant with the help of the equivalence between Seiberg-Witten and Gromov invariants.

Let us mention finally that it is easy to see that all symplectic forms on the k-fold blow-up of a rational or ruled manifold are deformation equivalent. However, it is nontrivial to decide whether two such forms are isotopic whenever they are cohomologous. Some progress with this uniqueness question has been made: see McDuff [17, 20] and Lalonde [5].

Here is a summary of the paper. In §3, we present the needed preliminaries about J-holomorphic curves: compatible almost complex structures, moduli spaces, compactness, regularity, positivity of intersection, and the adjunction formula (see [6] for a more complete summary). In §4, we prove Theorem 2.1 on the structure of rational and ruled symplectic 4-manifolds (except the statement on the uniqueness of the symplectic structure on $\mathbf{CP}^2$ which is proved in §5). In §5, we prove Theorems 2.3 and 2.1 as well as giving the proofs of some of the other fundamental results of Gromov, for example the fact that the group of compactly supported symplectomorphisms of $\mathbf{R}^4$ is contractible. In section §6, we prove Theorem 2.4, explaining in detail the symplectic and holomorphic parts of the argument, and showing how the rest of the proof can be reduced to the computation of the general wall-crossing formula for the Seiberg-Witten equations on ruled surfaces. Finally we sketch the proof of Theorem 2.2.

For the convenience of the reader, here is a list of contents.

3 Preliminaries

3.1 Summary of basic facts on J-holomorphic curves

All manifolds and maps will be assumed to be C^∞-smooth unless specific mention is made to the contrary.

An **almost complex structure** J on a manifold M is an automorphism $J : TM \to TM$ of the tangent bundle such that $J^2 = -\mathrm{Id}$. J is said to be

tamed by ω if

$$\omega(v, Jv) \quad > \quad 0, \quad \text{for all nonzero} \quad v \in TM,$$

and to be **compatible** with ω if in addition

$$\omega(Jv, Jw) \quad = \quad \omega(v, w), \quad \text{for all} \quad v, w \in TM.$$

As usual we write $\mathcal{J} = \mathcal{J}(M, \omega)$ for the set of all ω-compatible J (or the set of all ω-tame structures J, depending on the context). These spaces are both nonempty and contractible (because $\mathrm{Sp}(2n; \mathbf{R})$ retracts onto its maximal compact subgroup $U(n)$). (See [27, Chapter 2.5].) Given any $J \in \mathcal{J}(M, \omega)$, we write $c_1 \in H^2(M; \mathbf{Z})$ for the first Chern class of the complex vector bundle (TM, J). This is independent of the choice of $J \in \mathcal{J}(M, \omega)$ since the latter space is connected.

Definition 3.1 (Curves) A (parametrized) J-holomorphic curve in M is a map u from a Riemann surface (Σ, j) to M which satisfies the generalized (nonlinear elliptic) Cauchy–Riemann equation:

$$\bar{\partial}_J u = \tfrac{1}{2}(du \circ j - J \circ du) = 0.$$

The corresponding unparametrized curve $\mathrm{Im}\, u$ will often be denoted C. Thus C has real dimension 2 and complex dimension 1. When Σ is the Riemann sphere, the curve is often said to be **rational**. It is important to note that a J-holomorphic map $u : \Sigma \to M$ is either a **multiple cover**, i.e. it factors through a holomorphic map $\Sigma \to \Sigma'$ of degree > 1, or it is **somewhere injective** in the sense that there is a point $z \in \Sigma$ such that

$$du_z \neq 0, \quad u^{-1}(u(z) = \{z\}.$$

This is proved in McDuff [13] with more details given in [26, 2.3].

3.2 Local properties of J-curves

3.2.1 Singularities

We begin with a theorem essentially due to McDuff ([19, 22]) and improved by Sikorav in [37]. In particular, his argument works under much weaker smoothness assumptions on J.

Theorem 3.2 (Isolated singularities) *Let (M, J) be an almost complex manifold and u, u' two J-holomorphic maps to M defined on closed Riemann surfaces Σ, Σ'. Then the points where $u(z)$ coincides with $u'(z)$ are isolated. Further, the points where $du(z)$ vanishes are also isolated.*

Definition 3.3 Let $p \in (M, J)$ be the intersection of two J-holomorphic curves $C = u(\Sigma), C' = u'(\Sigma')$. By the previous theorem, we can define the contribution of p to the intersection of the two curves as follows: take small enough neighbourhoods U, U' of p in C, C' and perturb the interior of U, keeping its boundary fixed, so that:

(i) the perturbation of U, denoted say $\tilde{U}$, is C^0-small and stays disjoint from $\partial U'$, and

(ii) $\tilde{U}$ intersects U' transversally (at points which are regular both for $\tilde{U}$ and U').

Then define the contribution k_p of p to the intersection number of C and C' to equal the sum of the intersections of $\tilde{U}$ and U', each counted with the appropriate sign according to the given orientations of C, C', M.

3.2.2 Positivity of intersections and the adjunction formula

The positivity of intersections of two algebraic curves (which states that each point of intersection contributes positively) is a cornerstone of the algebraic geometry of surfaces. The previous theorem gives one hope that this principle extends to the almost complex case. And indeed it does, although the proof is much harder than in the integrable case. It is obvious that the sign of a point p of intersection of two J-curves is positive (and counts for $+1$) when the two curves are regular at p, and positivity is still quite easy to prove when at least one of the two curves is regular at p. It is more difficult when both curves are singular at p.

There is also a notion of positivity of intersection of a single curve: it says that each singular point of a complex curve gives a positive contribution to the self-intersection number of the curve. Again, this still holds in the almost complex case. The proof is quite elementary in the cases where the 1-jet of the singularity has the form (z^k, z^ℓ) with k, ℓ relatively prime, but is harder in the general case. In any case, part (i) of the following theorem was conjectured by Gromov in [2]. The full result (except with a C^0 rather than C^1 ϕ) was proved by McDuff in [16, 19, 22] by a topological argument based on perturbations. An improved and more analytical proof of everything except the last statement was given by Micallef and White in [29], by a method which works under considerably weaker smoothness assumptions on J.

Theorem 3.4 (Positivity) *(i) Let C, C' be two closed J-holomorphic curves. Then the contribution k_p of each point of intersection of C and C' to the intersection number $C \cdot C'$ is a strictly positive integer. It is equal to 1 if and only if C and C' are both regular at p and meet transversally at that point.*

(ii) For each singularity p of a J-holomorphic curve C, there is a neighbourhood U of p in M and a C^1-diffeomorphism ϕ from (U, p) to $(B, 0) \subset (\mathbf{C}^2, 0)$

which sends $C \cap U$ to an isolated singularity at 0 of a complex curve in the ball B. Moreover, there is a C^0-perturbation of J to J' with compact support inside U and a C^0-small isotopy of C to C' (or more precisely of its parametrisation) with compact support inside U such that the curve C' is immersed and J'-holomorphic.

Exercise 3.5 Prove the *adjunction formula*. This says that, if (M^4, J) is an almost complex manifold and $u : \Sigma \to M$ is a J-holomorphic curve which does not factorize through another Riemann surface by a multiple branched covering (that is: u is not a multiple covering), then the virtual genus of $C = \mathrm{Im}(u)$ defined by

$$g_v(C) = 1 + \frac{1}{2}(C \cdot C - c_1(C))$$

is always greater or equal to the genus of Σ, and it is equal to the genus of Σ if and only if u is an embedding. Note that this means that the total weight of singularities of a J-curve in (M^4, J) is a topological invariant. In particular, any J-curve homologous to an embedded J-curve of the same genus is also embedded.

Hint: suppose first that the curve is immersed and decompose the first Chern class of the ambient tangent bundle along the curve in the tangent and normal directions. Compute the self-intersection of the curve in terms of the self-intersection of the curve *inside* its normal bundle and of the number of self-intersection points of the curve (remember that these self-intersections are all positive!) Then deduce the general case from the immersed case using the last assertion of the theorem. For more details see [16].

3.3 Global geometry: moduli spaces and Gromov's compactness theorem

We now describe the global behaviour of the space of all J-holomorphic curves in a given homology class. It turns out that this space, as any solution space of an elliptic system of PDEs, is finite dimensional with dimension given by the Atiyah-Singer index theorem, at least when J is generic. We will see that, when this space is not empty, it is either compact or can be compactified by addition of what Gromov calls *cusp-curves*, which are the analogue of reducible curves in algebraic geometry. Actually, the picture is again very similar to the one in the integrable case. But the proofs are more delicate and rely on Riemannian estimates like the isoperimetric inequality and on properties of elliptic operators.

3.3.1 Fredholm framework

Let (M, ω) be a smooth compact symplectic manifold. As above, let $\mathcal{J}(M)$ be the Fréchet manifold of all almost complex structures which are tamed by