Low Rank Representations and Graphs for Sporadic Groups

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1.1 Introduction

Many interesting finite geometries, graphs and designs admit automorphism groups of low rank. In fact, it was a study of the rank 3 case which led to the discoveries and constructions of some of the sporadic simple groups (see [Gor82]). For several classification problems about graphs or designs, the case where the automorphism group is almost simple is of central importance, and many of the examples have a transitive automorphism group of low rank. This is the case, for example, for the classification problems of finite distance-transitive graphs [BCN89, PSY87], and of finite flag-transitive designs [BDD88, BDDKLS90].

This book presents a complete classification, up to conjugacy of the point stabilizers, of the faithful transitive permutation representations of rank at most 5 of the sporadic simple groups and their automorphism groups. These results, summarized in Chapter 5, filled a major gap in the existing classification results for finite, low rank, transitive permutation groups. For each representation classified, we also give the collapsed adjacency matrices (defined in Section 2.3) for all the associated orbital digraphs. We use these collapsed adjacency matrices to classify the vertex-transitive, distance-regular graphs for these low rank representations, and discover some new distance-regular graphs of diameter 2 (but of rank greater than 3) for the O'Nan group $O'N$, the Conway group Co_2, and the Fischer group Fi_{22}. We also classify the graphs of diameter at most 4 on which a sporadic simple group or its automorphism group acts distance-transitively. It turns out that all these graphs are well-known.

We have tried to give enough information so that the interested reader can duplicate most of our results, and study further the fascinating sporadic groups. In particular, we give presentations for most of the sporadic groups G having permutation representations of rank at most 5, together with sets of words generating the appropriate point stabilizers in G. This information allows the reader with access to a good coset enumeration program (such as those within MAGMA [CP95] and GAP [Sch95]) to reconstruct most of the representations studied in this book.

In the 1970s, R.T. Curtis determined many collapsed adjacency matrices for inclusion in the original Cambridge ATLAS, but these do not appear in the published ATLAS [CCNPW85]. In the early to mid 1980s, the primitive permutation representations of the nonabelian simple groups of order up to 10^6 (excluding the family $L_2(q)$) were analysed in detail from the point of view of cellular rings (or coherent configurations) by A.A. Ivanov, M.H. Klin and I.A. Faradžev [IKF82, IKF84] (see also [FIK90, FKM94]). As part of this analysis, these representations were explicitly constructed using the CoCo computer package [FK91], and all collapsed adjacency matrices for the orbital digraphs were determined. Furthermore, collapsed adjacency matrices have been constructed by others for certain specific orbital digraphs for sporadic groups (see [ILLSS95] and its references), but we have computed all the collapsed adjacency matrices in this book from scratch, using the methods we describe, except for two representations where explicit references are given.

Any permutation representation of rank at most 5 is multiplicity-free (that is, the sum of distinct complex irreducible representations), and for primitive permutation representations, the classification in this book has recently been extended in [ILLSS95], to give a complete classification of the primitive multiplicity-free permutation representations of the sporadic simple groups and their automorphism groups, together with a classification of the graphs Γ on which such a group acts primitively and distance-transitively. It is shown that for such a distance-transitive graph Γ, we have $\mathrm{diam}(\Gamma) \leq 4$, and so Γ appears in our classification. Even more recently, Breuer and Lux [BL96] have completed the classification of the imprimitive multiplicity-free permutation representations of the sporadic simple groups and their automorphism groups.

1.2 Transitive permutation groups, orbitals and ranks

The *symmetric group* on a set Ω is the group $\mathrm{Sym}(\Omega)$ of all permutations of Ω. If Ω is finite of cardinality n, then $\mathrm{Sym}(\Omega)$ is often denoted S_n. A *permutation group* G on a set Ω is a subgroup of $\mathrm{Sym}(\Omega)$, and G is said to be *transitive* on Ω if, for all $\alpha, \beta \in \Omega$, there is an element $g \in G$ such that the image α^g of α under g is equal to β. More generally, the *orbit* of G containing a point $\alpha \in \Omega$ is the set $\alpha^G := \{\alpha^g \mid g \in G\}$.

For the remainder of the section, let G be a transitive permutation group on a finite set Ω.

The permutation group G on Ω can also be regarded as a permutation group on $\Omega \times \Omega$ by defining

$$(\alpha, \beta)^g = (\alpha^g, \beta^g) \qquad (\alpha, \beta \in \Omega, g \in G).$$

The number of orbits of G on $\Omega \times \Omega$ is called the *rank* of G on Ω.

If α, β are distinct points of Ω, then the pairs (α, α) and (α, β) lie in different orbits of G on $\Omega \times \Omega$. Thus, for $|\Omega| > 1$, the rank of G is at least 2. A permutation group on Ω is said to be *2-transitive* (or *doubly transitive*) on Ω if it is transitive on the ordered pairs of distinct points of Ω. Thus, for $|\Omega| > 1$, the 2-transitive groups are precisely the permutation groups of rank 2. The classification of the finite 2-transitive groups was one of the first consequences for permutation groups of the finite simple group classification, and the problem of classifying finite permutation groups of low rank is a natural extension of this classification.

The orbits of G on $\Omega \times \Omega$ are called *orbitals*, and to each orbital E we associate the directed graph with vertex set Ω and edge set E, the so-called *orbital digraph* for E. It is easy to show that the orbitals for G are in one-to-one correspondence with the orbits on Ω of the *stabilizer* $G_\alpha := \{g \in G \mid \alpha^g = \alpha\}$ of a point $\alpha \in \Omega$. This correspondence maps an orbital E to the set of points $\{\beta \mid (\alpha, \beta) \in E\}$. The orbits of G_α on Ω are called *suborbits* of G, and their lengths are called the *subdegrees* of G.

If G has rank r, then a point stabilizer will have exactly r orbits on Ω, and we say that such a stabilizer is a *rank r subgroup* of G.

1.3 Permutation representations

Let G be a group and Ω a set. An *action* of G on Ω is a function which associates to every $\alpha \in \Omega$ and $g \in G$ an element α^g of Ω such that, for all $\alpha \in \Omega$ and all $g, h \in G$, $\alpha^1 = \alpha$ and $(\alpha^g)^h = \alpha^{gh}$. In a natural way, an action defines a *permutation representation* of G on Ω, which is a homomorphism φ from G into $\mathrm{Sym}(\Omega)$: simply define $(g)\varphi \in \mathrm{Sym}(\Omega)$ by $\alpha^{(g)\varphi} := \alpha^g$. Conversely, a permutation representation naturally defines an action of G on Ω, leading to a natural bijection between the actions of G on Ω and the permutation representations of G on Ω (see [NST94, pp. 30–32]). Note also that a permutation group H on Ω defines a natural representation (and action) of H on Ω, by defining the representation to be the identity map.

Most of the definitions of Section 1.2 apply to permutation representations by applying them to the permutation group which is the image of that representation. Thus, a permutation representation is said to be transitive if its image is transitive. Similarly, the orbits of a representation are those of its image and, if the representation is transitive, then its rank, orbitals, suborbits and subdegrees are those of its image. However, the point stabilizer $G_\alpha := \{g \in G \mid \alpha^g = \alpha\}$ for the representation may be a proper preimage of the point stabilizer for the permutation group image.

A permutation representation is said to be *faithful* if its kernel is the trivial group of order 1, in which case G is isomorphic to its permutation group image, and we are back to the case of permutation groups. In this book we study faithful representations of the sporadic simple groups and their automorphism groups. If a representation of a (sporadic) simple group is not faithful then clearly its image is the trivial group, and a non-faithful representation of the automorphism group of a sporadic simple group has an image of order 1 or 2 (as a sporadic simple group has index at most 2 in its automorphism group).

1.4 Permutational equivalence and permutational isomorphism

There are several slightly different concepts of equivalence, or isomorphism, for permutation representations and permutation groups (see [NST94, pp. 32–33]). Since an abstract group may be represented in many different ways as a permutation group, the notion of group isomorphism does not provide a sufficiently refined measure for distinguishing between different permutation representations and different permutation groups. The most general concept of permutational equivalence concerns different groups acting on different sets. A *permutational equivalence* of permutation representations of groups G, G^* acting on Ω, Ω^* respectively is a pair (θ, ϕ) of functions, where $\theta : \Omega \to \Omega^*$ is a bijection and $\phi : G \to G^*$ is an isomorphism, and

$$(\alpha^g)\theta = (\alpha\theta)^{g\phi}$$

for all $\alpha \in \Omega$ and all $g \in G$, and the representations (and actions) of G and G^* are said to be (permutationally) equivalent. Clearly θ induces a bijection between the sets of orbits of G, G^* in Ω, Ω^* respectively. Also, the restriction of ϕ to the stabilizer G_α of a point $\alpha \in \Omega$ is an isomorphism onto the stabilizer in G^* of the point $\alpha\theta \in \Omega^*$. Thus the equivalence induces bijections of the sets of orbits and point stabilizers of the two permutation representations. In the particular case of transitive representations of G, G^* on finite sets Ω, Ω^*, the permutational equivalence (θ, ϕ) preserves rank and subdegrees. Moreover, this equivalence induces a second equivalence $(\theta \times \theta, \phi)$ of the natural representations of G, G^* acting on $\Omega \times \Omega$ and $\Omega^* \times \Omega^*$ respectively (namely, by defining $(\alpha, \beta)(\theta \times \theta) := (\alpha\theta, \beta\theta)$ for all $(\alpha, \beta) \in \Omega \times \Omega$), such that $\theta \times \theta$ induces a bijection from the set of orbitals of G in $\Omega \times \Omega$ to the set of orbitals of G^* in $\Omega^* \times \Omega^*$, and preserves the isomorphism classes of the associated orbital digraphs.

If $G = G^*$ then the isomorphism ϕ is an automorphism of G. In the special case where ϕ is the identity map, the equivalence $(\theta, 1)$ is called a *permutational isomorphism.* Thus, roughly speaking, a permutational isomorphism amounts to a relabelling of the point set.

The notions of permutational equivalence and permutational isomorphism for permutation groups G, G^* on Ω, Ω^* respectively, are defined to be the same as these concepts for their natural representations. Note that the classification of faithful permutation representations up to permutational equivalence (respectively isomorphism) is the same as the classification of permutation groups up to permutational equivalence (respectively isomorphism).

In our subsequent discussion we use the following notation: for a group G, $\operatorname{Aut} G$ denotes the *automorphism group* of G, $\operatorname{Inn} G$ the group of *inner automorphisms* of G, and $\operatorname{Out} G := \operatorname{Aut} G / \operatorname{Inn} G$ is the *outer au-*

tomorphism group of G. Each element of $\operatorname{Aut} G \setminus \operatorname{Inn} G$ is called an *outer automorphism* of G.

Suppose that G has a transitive permutation representation on the set Ω, and choose $\alpha \in \Omega$. Then this representation is permutationally isomorphic to the representation of G, acting by right multiplication, on the right cosets of the point stabilizer G_α [NST94, Theorem 6.3]. If $\varphi \in \operatorname{Aut} G$, then G also has a transitive permutation representation, acting by right multiplication, on the set Ω^* of right cosets of the subgroup $K := (G_\alpha)\varphi$, and

$$\theta : \alpha^g \mapsto K(g\varphi) \quad (\alpha \in \Omega, g \in G)$$

is a well-defined bijection $\theta : \Omega \to \Omega^*$. Moreover the pair (θ, φ) is an equivalence between the permutation representations of G on Ω and on Ω^*. Of course (θ, φ) is by definition a permutational isomorphism if and only if φ is the identity. However the permutation representations of G on Ω and on Ω^* are permutationally isomorphic if and only if G_α and K are conjugate in G [NST94, Theorem 6.3 and Proposition 6.5]. We see from this discussion that, in general, two transitive representations of G are permutationally isomorphic if and only if a point stabilizer for one representation is in the same conjugacy class in G as a point stabilizer for the other representation. Moreover, there is a permutation representation of $\operatorname{Aut} G$ on the set of permutational isomorphism classes of transitive permutation representations of G such that $\operatorname{Inn} G$ is contained in the kernel. So in fact we have a permutation representation induced of $\operatorname{Out} G := \operatorname{Aut} G/\operatorname{Inn} G$ on these permutational isomorphism classes. The orbits of $\operatorname{Aut} G$ (and of $\operatorname{Out} G$) correspond to the permutational equivalence classes of transitive permutation representations of G. Thus the permutational isomorphism classes (respectively permutational equivalence classes) of transitive permutation representations of G are in one-to-one correspondence with the conjugacy classes of subgroups of G (respectively orbits of $\operatorname{Aut} G$, and hence of $\operatorname{Out} G$, on these conjugacy classes).

The classification of transitive permutation representations in this book is up to permutational isomorphism, which is the same as the classification up to conjugacy of the point stabilizers.

Two different permutational isomorphism classes of transitive representations correspond to the same permutational equivalence class if and only if there is an outer automorphism of G mapping one permutational isomorphism class to the other. In the case where G is a sporadic simple group, $|\mathrm{Out}\, G| \leq 2$. Hence, in this situation, an outer automorphism of G will interchange the two permutational isomorphism classes, and will also interchange the corresponding conjugacy classes of point stabilizers. We will point this out whenever it occurs.

1.5 Invariant partitions and primitivity

If G is a permutation group on a set Ω, then a partition P of Ω is said to be *G-invariant* (and G is said to *preserve* P) if the elements of G permute the blocks of P blockwise, that is, for $B \in P$ and $g \in G$, the set B^g is also a block of P. The blocks of a G-invariant partition are called *blocks of imprimitivity* for G. If G is transitive on Ω then all blocks of a G-invariant partition P of Ω have the same cardinality and G acts transitively on P. Moreover, every permutation group G on Ω preserves the two partitions $\{\Omega\}$ and $\{\{\alpha\} \mid \alpha \in \Omega\}$; these are called *trivial partitions* of Ω, and their blocks, Ω and $\{\alpha\}$ for $\alpha \in \Omega$, are called *trivial blocks of imprimitivity.* All other partitions of Ω are said to be *nontrivial.* A permutation group G is said to be *primitive* on Ω if G is transitive on Ω and the only G-invariant partitions of Ω are the trivial ones. Also G is said to be *imprimitive* on Ω if G is transitive on Ω and G preserves some nontrivial partition of Ω.

1.6 The O'Nan-Scott theorem for finite primitive permutation groups

It is not difficult to see that the set of orbits of a normal subgroup of a transitive permutation group G on Ω is a G-invariant partition of Ω. Thus each nontrivial normal subgroup of a primitive permutation group is transitive. In particular, for finite primitive permutation groups G on Ω the *socle* of G, *soc*(G), which is the product of its minimal normal subgroups, is transitive on Ω. Several different types of finite primitive permutation groups have been identified in the O'Nan-Scott Theorem ([Sco80, AS85] or see [LPS88]) and are described according to the structure and permutation action of their socles.

A finite primitive permutation group G has at most two minimal normal subgroups, and if M, N are distinct minimal normal subgroups of G, then $M \cong N$, M and N are nonabelian, and both act regularly on Ω

(see [Sco80] or [LPS88]). (A permutation group on Ω is *regular* if it is transitive, and only the identity element fixes a point of Ω.)

For most of the types of finite primitive groups, the socle is the unique minimal normal subgroup, and for all types the socle is a direct product of isomorphic simple groups. The types of finite primitive permutation groups are described in [LPS88] as follows. Let G be a primitive permutation group on a finite set Ω, and let $N := soc(G)$. Then $N = T^k$ for some simple group T and positive integer k, and one of the following holds.

Affine type. Here $N = Z_p^k$ (p a prime) is elementary abelian, N is the unique minimal normal subgroup of G, N is regular on Ω, and Ω can be identified with a finite vector space V in such a way that N is the group of translations of V and G is a subgroup of the group $AGL(V)$ of affine transformations of V.

Amost simple type. The socle $N = T$ is a nonabelian simple group ($k = 1$), so $T \le G \le \operatorname{Aut} T$, that is, G is an ***almost simple group***. Also $T_\alpha \neq 1$.

For the remaining types $N = T^k$ with $k \ge 2$ and T a nonabelian simple group.

Simple diagonal type. Here G is a subgroup of the group

$$\begin{aligned} W \;:=\; & \{(a_1, \ldots, a_k).\pi \mid a_i \in \operatorname{Aut} T, \pi \in S_k, \\ & a_i \equiv a_j \pmod{\operatorname{Inn} T} \text{ for all } i, j\}, \end{aligned}$$

where $\pi \in S_k$ permutes the components a_i naturally. With the obvious multiplication, W is a group with socle $N = T^k$, and $W = N.(\operatorname{Out} T \times S_k)$, a (not necessarily split) extension of N by $\operatorname{Out} T \times S_k$. The action of W on Ω is equivalent to its action by right multiplication on the set of right cosets of its subgroup

$$W_\alpha := \{(a, \ldots, a).\pi \mid a \in \operatorname{Aut} T, \pi \in S_k\} \cong \operatorname{Aut} T \times S_k.$$

The group G must contain N, and $N_\alpha = \{(a, \ldots, a) \mid a \in T\}$ is a diagonal subgroup of N, hence the name 'diagonal type'.

Product type. For this type, G is a subgroup of a wreath product $W := H \operatorname{wr} S_l$ in product action on $\Omega = \Lambda^l$, where $l \ge 2$ and l divides k, H is a primitive permutation group on Λ, $soc(H) \cong T^{k/l}$, and $N = soc(W) = soc(H)^l$ is contained in G. The group H is of either almost simple or simple diagonal type.

Twisted wreath type. For this type, $G = T \operatorname{twr}_{\varphi} P = T^k.P$ is a twisted wreath product, where $P \leq S_k$, and N is regular on Ω.

More information about the structure of these groups can be found in [AS85, Sco80, LPS88].

1.7 Existing classifications of low rank primitive groups

Long before the description of finite primitive permutation groups that we find in the O'Nan-Scott Theorem had been written down, W. Burnside [Bur11, Section 154] proved that a finite 2-transitive group is of either affine or almost simple type. In fact, the minimum ranks for finite primitive groups of the other types tend to be higher than those for primitive groups of affine or almost simple type, and it follows from the O'Nan-Scott Theorem that the finite primitive groups of rank at most 5 are essentially known once the almost simple ones and the affine ones have been classified (see [Cuy89]). According to the finite simple group classification a nonabelian finite simple group T is either an alternating group, a group of Lie type, or one of the 26 sporadic simple groups (see [Gor82]). Thus the socle T of an almost simple group G is a simple group of one of these types.

The finite 2-transitive groups have been completely classified using the finite simple group classification, and this result is the culmination of the work of many people. The 2-transitive representations of the finite symmetric and alternating groups were classified by E. Maillet [Mai1895] in 1895. Those of the finite almost simple groups of Lie type were determined by C.W. Curtis, W.M. Kantor and G.M. Seitz [CKS76] in 1976, and the classification of the 2-transitive groups of almost simple type was completed and announced by P.J. Cameron [Cam81] in 1981 as a consequence of the finite simple group classification. The finite soluble 2-transitive groups were classified by B. Huppert [Hup57] in 1957; the major part of the classification of the finite insoluble 2-transitive groups of affine type was done by C. Hering [Her74, Her85], and a complete and independent proof of the classification of finite 2-transitive groups of affine type was given by M.W. Liebeck [Lie87, Appendix 1].

A great deal of effort has gone into understanding low rank primitive permutation groups, in particular those of rank at most 5. It follows from the O'Nan-Scott Theorem (see [Cuy89, Corollary 2.2]) that, if G

is primitive of rank at most 5 on a finite set Ω, then either G is of affine or almost simple type, or G is a subgroup of a wreath product $H \operatorname{wr} S_k$ in product action on $\Omega = \Lambda^k$, where $k \in \{2,3,4\}$ and H is an almost simple 2-transitive permutation group on Λ, or G has simple diagonal type with socle isomorphic to $L_2(q) \times L_2(q)$ for some $q \in \{5,7,8,9\}$. Thus a classification of primitive permutation groups of rank up to 5 is reduced to a classification of those of affine or almost simple type.

We consider the almost simple case first. In 1972 E.E. Bannai [Ban72] classified all primitive permutation representations of rank at most 5 of the finite alternating and symmetric groups. In 1982 W.M. Kantor and R.A. Liebler [KL82] classified the primitive rank 3 representations of the classical groups (see also [Sei74]). In 1986 M.W. Liebeck and J. Saxl [LS86] found all the primitive rank 3 representations of the exceptional simple groups of Lie type, and A. Brouwer, R.A. Wilson and L.H. Soicher (see [LS86]) determined those of the sporadic simple groups, thereby completing the classification of almost simple primitive rank 3 groups. In 1989, H. Cuypers [Cuy89] completed the classification of all primitive representations of rank at most 5 of all finite almost simple groups of Lie type. Part of the purpose of this book is to complete the classification of the almost simple primitive groups of rank at most 5 by classifying all such representations of the sporadic almost simple groups. Note that the sporadic almost simple groups are the sporadic simple groups and their automorphism groups, since a sporadic simple group has index at most 2 in its automorphism group.

Finite soluble primitive groups of rank 3 are primitive groups of affine type and were classified by D.A. Foulser [Fou69] in 1969. The classification of all primitive rank 3 groups of affine type was completed by M.W. Liebeck [Lie87] in 1987. From these results, and the results presented in this book, it follows that to complete the classification of finite primitive permutation groups of rank at most 5, only the affine primitive groups of rank 4 and 5 remain to be classified.

1.8 Low rank sporadic classification

In this book we classify all (primitive and imprimitive) faithful transitive permutation representations of rank at most 5 of the sporadic simple groups and their automorphism groups. The rank 2 case is included for completeness, and the rank 3 case expands the list in [LS86] by classifying all imprimitive rank 3 representations of these groups.

We also provide detailed information about the digraphs on which the permutation groups we describe act vertex-transitively. Background about such graphs is given in Chapter 2, and a discussion of the methods used in our investigations is in Chapter 3. Chapter 4 contains the main body of our work, with the description of the representations and digraphs for the individual sporadic groups, together with many presentations, and Chapter 5 summarizes the representations and distance-regular graphs classified.