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1
Preliminaries

We assume that the reader is familiar with the fundamental notions of differential geometry. From a large number of mathematical texts we suggest the books of Helgason (1962), Bishop and Crittendon (1964), Kobayashi and Nomizu (1969), Matsushima (1972), Westenholz (1978), Spivak (1979), Choquet–Bruhat *et al.* (1982) and Willmore (1993) for comprehensive introductions into the subject. Concise accounts, designed to meet the needs of a relativist, can be found in Hawking and Ellis (1973), Kramer *et al.* (1980), Chandrasekhar (1983), Straumann (1984) and Wald (1984). The reader is referred to these books for introductions into the concepts of manifolds, tensor fields, connections and curvature. Since we intend to use an efficient formalism, we shall start this text with a brief review of the basic properties of differential forms. Before doing so, we fix some notations and conventions.

1.1 Conventions

Throughout this text, spacetime is denoted by $(M, \boldsymbol{g})$, where M is a 4–dimensional differentiable manifold endowed with a pseudo–Riemannian metric $\boldsymbol{g} = g_{\mu\nu}\, \theta^\mu \otimes \theta^\nu$ with signature $(-,+,+,+)$. (In order to avoid confusion, we occasionally write ${}^{(4)}\boldsymbol{g}$ instead of $\boldsymbol{g}$.) Greek indices label spacetime components of tensor fields, whereas Latin indices usually refer to components of lower–dimensional quantities.

The tangent space at a point $p \in M$ is denoted by $T_p(M)$ and the set of C^∞ vector fields X by $\mathcal{X}(M)$. We use the symbol ∇ for the unique metric, torsion–free affine connection on M, assigning the vector field $\nabla_X Y \in \mathcal{X}(M)$ to every pair of C^∞ vector fields $X, Y \in \mathcal{X}(M)$.

The curvature tensor maps the triple of fields $X, Y, Z \in \mathcal{X}(M)$

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to the vector field $R(X,Y)Z \in \mathcal{X}(M)$. In terms of the connection ∇, the latter is defined by

$$R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z \,. \tag{1.1}$$

The components of the Ricci tensor are obtained by the contraction

$$R_{\mu\nu} = R_{\mu\alpha\nu}{}^{\alpha} \,, \tag{1.2}$$

where the components of the Riemann tensor are found from eq. (1.1). In terms of Christoffel symbols one has

$$R_{\mu\beta\nu}{}^{\alpha} = \partial_\beta \Gamma^\alpha_{\mu\nu} - \partial_\mu \Gamma^\alpha_{\beta\nu} + \Gamma^\sigma_{\mu\nu}\Gamma^\alpha_{\beta\sigma} - \Gamma^\sigma_{\beta\nu}\Gamma^\alpha_{\mu\sigma} \,. \tag{1.3}$$

The commutation relations for the second covariant derivatives of a vector field $X \in \mathcal{X}(M)$ are

$$[\nabla_\nu\nabla_\mu - \nabla_\mu\nabla_\nu]\, X^\beta = R_{\mu\nu\alpha}{}^{\beta} X^\alpha \,. \tag{1.4}$$

Let $\boldsymbol{T} = T_{\mu\nu}\theta^\mu \otimes \theta^\nu$ and $R = R_{\mu\nu}g^{\mu\nu}$ denote the stress–energy tensor of the matter fields and the Ricci scalar, respectively. The metric fields $g_{\mu\nu}$ are subject to Einstein's field equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 8\pi\, G\, T_{\mu\nu} \,, \tag{1.5}$$

which form a system of ten nonlinear, second order partial differential equations. The speed of light is set equal to 1 throughout this text.

1.2 Differential forms

Differential forms arise naturally in Riemannian geometry. Numerous formulae in general relativity are most efficiently obtained by deriving them within the framework of the exterior calculus. For instance, it is often easier to solve Cartan's structure equations than to compute the Riemann tensor from the Christoffel symbols. It is not our intention to provide an introduction into the exterior calculus in this section. Instead, we give only a brief account of the basic concepts and fix the conventions which we will need later. The reader who is not familiar with the subject should, for instance, consult Willmore (1993) for a concise introduction.

Consider an n–dimensional, orientable (pseudo–)Riemannian manifold $(M, \boldsymbol{g})$. Let $\Lambda(M) = \oplus_0^n \Lambda_p(M)$ denote the exterior algebra of differential forms on M. In a positively oriented local

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coordinate system $\{x^i\}$ $(i = 1...n)$ one has the representation

$$\alpha = \tfrac{1}{p!}\, \alpha_{\mu_1 \dots \mu_p}\, dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \,, \tag{1.6}$$

$$\eta = \tfrac{1}{n!}\, \eta_{\mu_1 \dots \mu_n}\, dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n} = \sqrt{|g|}\, dx^1 \wedge \dots \wedge dx^n \,, \tag{1.7}$$

for an arbitrary p–form $\alpha \in \Lambda_p(M)$ and for the volume–form $\eta \in \Lambda_n(M)$, respectively. In terms of the determinant $g = \det(g_{\mu\nu})$ of the metric, the components of the volume–form are

$$\eta_{\mu_1 \dots \mu_n} = \sqrt{|g|}\, \varepsilon_{\mu_1 \dots \mu_n} \,, \tag{1.8}$$

where $\varepsilon_{\mu_1 \dots \mu_n} = 1\,(-1)$ if $(\mu_1, \dots, \mu_n)$ is an even (odd) permutation of $(1, \dots, n)$, and $\varepsilon_{\mu_1 \dots \mu_n} = 0$ otherwise.

Let us denote the interior multiplication of a p–form $\alpha \in \Lambda_p(M)$ by a vector field $X \in \mathcal{X}(M)$ by $i_X \alpha$, and let $d\alpha$ be the exterior derivative of α. The endomorphisms i_X and d are anti–derivatives $\Lambda(M) \rightarrow \Lambda(M)$ of degree -1 and 1, respectively:

$$(i_X\, \alpha)(X_1, \dots X_{p-1}) = \alpha(X, X_1, \dots X_{p-1}) \quad \in \Lambda_{p-1}(M) \,, \tag{1.9}$$

$$d\alpha = d\,(\alpha_{\mu_1 \dots \mu_p}) \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \quad \in \Lambda_{p+1}(M) \,. \tag{1.10}$$

Note that the definition of $d\alpha$ is independent of the coordinate system. Also recall that $d\alpha = 0$ for $\alpha \in \Lambda_n(M)$ and $i_X \alpha = 0$ for $\alpha \in \Lambda_0(M)$. In addition, both i_X and d give zero when repeatedly applied to any p–form:

$$i_X \circ i_X\, \alpha = 0 \,, \quad d \circ d\, \alpha = 0 \quad \text{for any } \alpha \in \Lambda_p(M). \tag{1.11}$$

A p–form α is called *closed* if $d\alpha = 0$, and *exact* if there exists a $(p-1)$–form β such that $\alpha = d\beta$. Clearly every exact form is closed whereas, in general, the converse statement holds only locally. More precisely, Poincaré's lemma states that in a star–shaped domain every closed form is exact.

To every p–form α one can assign its Hodge dual, the $(n-p)$–form $*\alpha$, defined such that

$$\alpha \wedge \beta = (-1)^s\, (*\alpha \,|\, \beta)\, \eta \,, \tag{1.12}$$

for all $\beta \in \Lambda_{n-p}(M)$, where s denotes the number of negative eigenvalues of the metric. (Here we have used the natural extension of the inner product to each $\Lambda_p(M)$.) The components of $*\alpha$ are

$$(*\alpha)_{\mu_{p+1} \dots \mu_n} = \frac{1}{p!}\, \eta_{\mu_1 \dots \mu_n}\, \alpha^{\mu_1 \dots \mu_p} \,, \tag{1.13}$$

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where $\alpha^{\mu_1 \cdots \mu_p} = g^{\mu_1 \nu_1} \cdot \ldots \cdot g^{\mu_p \nu_p} \, \alpha_{\nu_1 \ldots \nu_p}$. The inverse and the square of the Hodge dual of a p–form are given by

$$*^{-1} \, = \, (-1)^{p(n-p)+s} \, * \, , \qquad *^2 \, = \, (-1)^{p(n-p)+s} \, . \tag{1.14}$$

Hence, for arbitrary p–forms α, $\beta \in \Lambda_p(M)$, one has the identity

$$(\alpha|\beta) \, \eta \, = \, \alpha \wedge *\beta \, = \, \beta \wedge *\alpha \, = \, (-1)^s \, (*\alpha \, | \, *\beta) \, \eta \, , \tag{1.15}$$

since

$$*\, 1 \, = \, \eta \, , \qquad * \eta \, = \, (-1)^s \, . \tag{1.16}$$

To every p–form α one can also assign the $(p-1)$–form $d^\dagger \alpha \in \Lambda_{p-1}(M)$. In terms of the exterior derivative and the Hodge dual, the latter is defined by

$$d^\dagger \alpha \, = \, -(-1)^{n(p+1)+s} \, * d * \alpha \, . \tag{1.17}$$

The component expression for $d^\dagger \alpha$ becomes

$$(d^\dagger \alpha)^{\mu_1 \cdots \mu_{p-1}} \, = \, - \frac{1}{\sqrt{|g|}} \, \partial_\mu \left(\sqrt{|g|} \, \alpha^{\mu \, \mu_1 \cdots \mu_{p-1}} \right) . \tag{1.18}$$

The operator $d^\dagger : \Lambda_p(M) \to \Lambda_{p-1}(M)$ is called the co–derivative. The sign convention in the above definition is chosen such that d and $d^\dagger$ are formal adjoints of each other with respect to the inner product

$$\langle \, \cdot \, , \, \cdot \, \rangle \, = \, \int_M (\, \cdot \, | \, \cdot \,) \, \eta \, , \tag{1.19}$$

that is, such that $\langle \, d\alpha \, , \, \beta \, \rangle = \langle \, \alpha \, , \, d^\dagger \beta \, \rangle$ for an orientable manifold M and $\alpha \in \Lambda_{p-1}(M)$, $\beta \in \Lambda_p(M)$.

A p–form α is said to be harmonic if both $d^\dagger \alpha$ and $d\alpha$ vanish and hence $\Delta \alpha = 0$, where, for arbitrary p–forms, the Laplacian is defined by

$$\Delta \, = \, - \left[d^\dagger d \, + \, d \, d^\dagger \right] . \tag{1.20}$$

The sign convention in this definition is such that we obtain the usual coordinate expression

$$\Delta f \, = \, \frac{1}{\sqrt{|g|}} \, \partial_\nu (\sqrt{|g|} \, g^{\mu\nu} \, \partial_\mu f) \tag{1.21}$$

for the Laplacian of a function f. (In the case of a pseudo–Riemannian metric the operator defined in eq. (1.20) is also called the d'Alembertian and is denoted by "$\Box$". Throughout this text we shall use the symbol Δ and, for any signature of the metric, refer to it as the Laplacian.)

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Let us now restrict ourselves to the $3+1$–dimensional case. For $n = 4$ and $\mathrm{sig}(g) = (-,+,+,+)$ we find

$$*^{-1} \;=\; -(-1)^p *\,, \quad *^2 \;=\; -(-1)^p\,, \quad d^\dagger \;=\; *d*\,. \tag{1.22}$$

In particular, the co–derivative of the product of a function $f \in \Lambda_0(M)$ with a 1–form $\alpha \in \Lambda_1(M)$ becomes

$$d^\dagger(f\,\alpha) \;=\; f\,d^\dagger\alpha \;-\; (df|\alpha)\,. \tag{1.23}$$

In the following we shall often take advantage of the fact that the interior multiplication can be expressed in terms of the Hodge dual and the exterior product as

$$i_X\alpha \;=\; -*\,(X \wedge *\alpha)\,, \quad i_X * \alpha \;=\; *(\alpha \wedge X)\,, \tag{1.24}$$

where $\alpha \in \Lambda_p(M)$. Here we have used the symbol X for both the vector field $X \in \mathcal{X}(M)$ and the 1–form $X \in \Lambda_1(M)$ associated with it. As an application, we obtain the identity

$$\begin{aligned} (X|Y)\,(\alpha|\alpha) \;&=\; (X \wedge \alpha|Y \wedge \alpha) \;-\; (X \wedge *\alpha|Y \wedge *\alpha) \\ &=\; (i_X\alpha|i_Y\alpha) \;-\; (i_X * \alpha|i_Y * \alpha)\,, \end{aligned} \tag{1.25}$$

which holds for arbitrary 1–forms $X, Y \in \Lambda_1(M)$ and arbitrary p–forms $\alpha \in \Lambda_p(M)$, since

$$\begin{aligned} (X \wedge \alpha|Y \wedge \alpha)\eta \;&=\; -i_Y * \alpha \wedge (X \wedge \alpha) = (-1)^p * \alpha \wedge i_Y(X \wedge \alpha) \\ &=\; (-1)^p\,[\,(X|Y)(*\alpha \wedge \alpha) - *\alpha \wedge X \wedge i_Y\alpha\,] \\ &=\; [\,(X|Y)\,(\alpha|\alpha) + (X \wedge *\alpha|Y \wedge *\alpha)\,]\,\eta\,. \end{aligned}$$

This establishes the first line in eq. (1.25). Applying eqs. (1.15) and (1.24), we also obtain the second part of the identity since $(X\wedge\alpha|Y\wedge\alpha) = -(i_X*\alpha|i_Y*\alpha)$ and $(X\wedge*\alpha|Y\wedge*\alpha) = -(i_X\alpha|i_Y\alpha)$. For $\alpha \in \Lambda_1(M)$, eq. (1.25) yields the familiar formula

$$(X \wedge \alpha|Y \wedge \alpha) \;=\; (X|Y)\,(\alpha|\alpha) \;-\; (X|\alpha)\,(Y|\alpha)\,. \tag{1.26}$$

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2
Spacetimes admitting Killing fields

Einstein's field equations form a set of nonlinear, coupled partial differential equations. In spite of this, it is still sometimes possible to find exact solutions in a systematic way by considering spacetimes with symmetries. Since the laws of general relativity are covariant with respect to diffeomorphisms, the corresponding reduction of the field equations must be performed in a coordinate–independent way. This is achieved by using the concept of Killing vector fields. The existence of Killing fields reflects the symmetries of a spacetime in a coordinate–invariant manner.

A spacetime $(M, \boldsymbol{g})$ admitting a Killing field gives rise to an invariantly defined 3–manifold Σ. However, Σ is only a hypersurface of $(M, \boldsymbol{g})$ if it is orthogonal to the Killing trajectories. In general, Σ must be considered to be a quotient space M/G rather than a subspace of M. (Here G is the 1–dimensional group generated by the Killing field.) The projection formalism for M/G was developed by Geroch (1971, 1972a), based on earlier work by Ehlers (see also Kramer *et al.* 1980). The invariant quantities which play a leading role are the twist and the norm of the Killing field.

In the first section of this chapter we compile some basic properties of Killing fields. The twist, the norm and the Ricci 1–form assigned to a Killing field are introduced in the second section. Using these quantities, we then give the complete set of reduction formulae for the Ricci tensor.

In the third section we apply these formulae to vacuum spacetimes. In particular, we introduce the vacuum Ernst potential and derive the entire set of field equations from a variational principle. As we shall see later, these equations reduce to the ordinary Ernst equations if spacetime admits two Killing fields satisfying the Frobenius integrability conditions.

The fourth and fifth sections are devoted to stationary and

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static spacetimes, respectively. We recall the notions of Ricci–staticity and metric staticity and discuss the relationship between the two concepts. In the sixth section we derive some formulae for static spacetimes admitting a foliation by regular 2–surfaces. These will be relevant to later applications - especially to the original proof of the Israel theorem.

Spacetimes admitting two Killing fields are discussed in the last section of this chapter. In an asymptotically flat spacetime, the Killing fields generating the stationary and axisymmetric isometries form an Abelian group. This leads to the notion of stationary *and* axisymmetric spacetimes. We shall introduce the concepts of Ricci–circularity and metric circularity and discuss the integrability conditions for the Killing fields within these terms. In particular, we give a simple proof of the general circularity theorem, establishing the integrability conditions for Ricci–circular spacetimes. Some implications of the Frobenius conditions for the Killing 2–form assigned to a stationary and axisymmetric spacetime are discussed at the end of this chapter.

2.1 Killing fields

Consider the 1–parameter group of diffeomorphisms $\phi : M \to M$ generated by a vector field $X \in \mathcal{X}(M)$. A tensor field is invariant under ϕ if its Lie derivative with respect to X vanishes.

Definition 2.1 *A vector field* $X \in \mathcal{X}(M)$ *satisfying*

$$L_X \boldsymbol{g} = 0 \tag{2.1}$$

is called a Killing field.

We recall that the 1–parameter group of point transformations corresponding to a Killing field is an isometry.

In order to avoid an unnecessarily complicated notation, we shall use the same symbol for vector fields and their associated 1–forms. The following relations, which hold for arbitrary p–forms $\alpha \in \Lambda_p(M)$ and Killing fields (1–forms) K, turn out to be useful:

$$d^\dagger K = 0\,, \tag{2.2}$$

$$L_K * \alpha = * L_K \alpha\,, \tag{2.3}$$

$$d^\dagger(K \wedge \alpha) + K \wedge d^\dagger \alpha = -L_K \alpha\,. \tag{2.4}$$

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The first equation is the contracted dual of the Killing equation for vector fields, $\nabla^\mu K^\nu + \nabla^\nu K^\mu = 0$. The commutation property of the Hodge dual with the Lie derivative with respect to a Killing field is easily verified for arbitrary p–forms. In order to derive the last relation, we apply the operator identity

$$L_K = d \circ i_K + i_K \circ d \tag{2.5}$$

on the $(4-p)$–form $*\alpha$ and use eqs. (1.24) and (1.22):

$$\begin{aligned} L_K * \alpha &= - * (K \wedge *d * \alpha) + (-1)^p \, d * (K \wedge \alpha) \\ &= - * (K \wedge d^\dagger \alpha) - * d^\dagger (K \wedge \alpha) \, . \end{aligned}$$

Taking the dual of this expression and using eq. (2.3) yields the desired result.

Definition 2.2 *The* 1*–form* $R(X)$ *assigned to a vector field* $X \in \mathcal{X}(M)$ *with components*

$$R(X)_\mu = R_{\mu\nu} X^\nu \tag{2.6}$$

is called the Ricci 1*–form with respect to* X.

Contracting expression (1.4) and using the Killing equation, we see that the definition (1.20) of the Laplacian immediately yields the Ricci identity for a Killing field (1–form) K,

$$\Delta K = -2 \, R(K) \, . \tag{2.7}$$

We conclude this section by giving Stokes' theorem in the presence of a Killing field. The 1–dimensional group of isometries generated by the Killing field K give rise to an invariantly defined 3–manifold Σ (see Geroch 1971). Consider a 1–form α which is invariant under the action of this group, $L_K \alpha = 0$. Integrating the dual of the identity (2.4) and using Stokes' theorem ($\int d\cdot = \int_\partial \cdot$), we find for $\alpha \in \Lambda_1(M)$

$$\int_{\partial\Sigma} *(K \wedge \alpha) = - \int_\Sigma (d^\dagger \alpha) \; i_K \eta \, , \quad \text{if } L_K \alpha = 0 \, , \tag{2.8}$$

where $i_K \eta = *K$. This form of Stokes' theorem turns out be useful, for instance, in deriving integral identities relating the mass and the total charge of an electrovac spacetime (see chapter 8).

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2.2 Basic identities

In this section we establish a set of differential identities between the Ricci 1–form, the twist and the norm of a Killing field. As we shall see later, these relations reduce to the Ernst equations for vacuum and electrovac spacetimes. In addition, they turn out to be the key identities for the proof of the staticity and circularity theorems.

Definition 2.3 *Let K be a Killing field (1–form). The function $N \in \Lambda_0(M)$ and the 1–form $\omega \in \Lambda_1(M)$,*

$$N = (K|K), \quad \omega = \frac{1}{2} * (K \wedge dK) \tag{2.9}$$

are called the norm and the twist (rotation–form) associated with K. In the domain of $(M, \boldsymbol{g})$ where N is positive (negative, zero) the field K is said to be spacelike (timelike, null).

By virtue of eq. (1.24) and the above definition for ω, we have $-2 * (K \wedge \omega) = 2i_K * \omega = i_K(K \wedge dK)$. Now using the fact that $i_K dK = -di_K K = -dN$ (since $L_K K = 0$), we obtain the expression

$$-dK = \frac{1}{N}[2 * (K \wedge \omega) + K \wedge dN] \tag{2.10}$$

for the derivative of the Killing form in terms of its twist and norm. Taking advantage of this formula, it is also not hard to verify the identity

$$N (dK|dK) = (dN|dN) - 4 (\omega|\omega). \tag{2.11}$$

(First, the cross–terms do not contribute, since they are proportional to $K \wedge K$. Secondly, eq. (1.15) yields $(*[K \wedge \omega] \,|\, *[K \wedge \omega]) = -(K \wedge \omega | K \wedge \omega) = -N(\omega|\omega)$, since $(K|\omega) = 0$. Finally, eq. (1.26) implies $(K \wedge dN | K \wedge dN) = N(dN|dN)$, since $(K|dN) = L_K N = 0$.)

It is worthwhile pointing out that the Killing property of K has not been used so far. Hence, the above identities between the norm and twist hold for arbitrary vector fields (1–forms). However, in the remainder of this chapter, K is assumed to be a Killing field.

We now derive the formulae for the derivative and the co–derivative of the twist ω. We first note that both N and ω have vanishing Lie derivatives with respect to K, $L_K N = L_K \omega = 0$.

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Hence, considering $\alpha = \omega/N^2$ in the general identity (2.4) and using the immediate consequence

$$-2 * (K \wedge \frac{\omega}{N^2}) = d(\frac{K}{N}) \tag{2.12}$$

of equation (2.10), we obtain

$$K \wedge d^\dagger (\frac{\omega}{N^2}) = -d^\dagger (K \wedge \frac{\omega}{N^2}) = \frac{1}{2} d^\dagger * d(\frac{K}{N}) = 0,$$

since $d^\dagger * d = \pm * d^2 = 0$. (Here we have used the fact that K is a Killing field, $d^\dagger K = 0$.) Thus, the norm and twist satisfy the differential identity

$$d^\dagger (\frac{\omega}{N^2}) = 0 \quad \text{or} \quad d^\dagger \omega = -2 \frac{(\omega|dN)}{N} \tag{2.13}$$

(i.e., $N\nabla^\mu \omega_\mu = +2\omega_\mu \nabla^\mu N$). Again using the identity (2.4) and $\Delta K = -d^\dagger dK = 0$, the exterior derivative of ω is obtained as follows:

$$2\, d\omega = - * d^\dagger (K \wedge dK) = - * (K \wedge \Delta K). \tag{2.14}$$

In order to find an expression also involving the inner product of K with ΔK, we compute the Laplacian of N. By definition, we have $\Delta N = -d^\dagger dN$. Now using $dN = -i_K dK = *(K \wedge *dK)$ we obtain

$$\begin{aligned} -\Delta N &= *d(K \wedge *dK) = [(dK|dK) + (K|\Delta K)] * \eta \\ &= \frac{1}{N}[4(\omega|\omega) - (dN|dN)] - (K|\Delta K), \end{aligned} \tag{2.15}$$

where we have also used $*\eta = -1$ and the relation (2.11) to substitute the quadratic term in dK. Taking advantage of the Ricci identity (2.7) enables us to replace the Laplacian of K in eqs. (2.14) and (2.15) by $-2R(K)$. In conclusion, we have established the following result:

Proposition 2.4 *The exterior derivative $d\omega$, the co–derivative $d^\dagger\omega$ and the square $(\omega|\omega)$ of the twist–form are given in terms of the Ricci 1–form $R(K)$ by*

$$d\omega = *(K \wedge R(K)), \tag{2.16}$$

$$d^\dagger \omega = -2 N^{-1} (\omega \,|\, dN), \tag{2.17}$$

$$(\omega|\omega) = \frac{1}{4}[(dN|dN) - N\Delta N - 2N R(K,K)]. \tag{2.18}$$

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