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Young Tableaux

With Applications to Representation Theory and Geometry

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Preface

The aim of this book is to develop the combinatorics of Young tableaux, and to see them in action in the algebra of symmetric functions, representations of the symmetric and general linear groups, and the geometry of flag varieties. There are three parts: Part I develops the basic combinatorics of Young tableaux; Part II applies this to representation theory; and Part III applies the first two parts to geometry.

Part I is a combinatorial study of some remarkable constructions one can make on Young tableaux, each of which can be used to make the set of tableaux into a monoid: the Schensted "bumping" algorithm and the Schützenberger "sliding" algorithm; the relations with words developed by Knuth and Lascoux and Schützenberger, and the Robinson–Schensted–Knuth correspondence between matrices with nonnegative integer entries and pairs of tableaux on the same shape. These constructions are used for the combinatorial version of the Littlewood–Richardson rule, and for counting the numbers of tableaux of various types.

One theme of Parts II and III is the ubiquity of certain basic quadratic equations that appear in constructions of representations of S_n and $GL_m\mathbb{C}$, as well as defining equations for Grassmannians and flag varieties. The basic linear algebra behind this, which is valid over any commutative ring, is explained in Chapter 8. Part III contains, in addition to the basic Schubert calculus on a Grassmannian, a last chapter working out the Schubert calculus on a flag manifold; here the geometry of flag varieties is used to construct the Schubert polynomials of Lascoux and Schützenberger.

There are two appendices. Appendix A contains some of the many variations that are possible on the combinatorics of Part I, but which are not needed in the rest of the text. Appendix B contains the topology needed to assign a cohomology class to a subvariety of a nonsingular projective variety, and prove the basic properties of this construction that are needed for a study of flag manifolds and their Schubert subvarieties.

There are numerous exercises throughout. Except for a few numerical verifications, we have included answers, or at least hints or references, at the end of the text.

Our main aim has been to make the various main approaches to the calculus of tableaux, and their applications, accessible to nonexperts. In particular, an important goal has been to make the important work of Lascoux and Schützenberger more accessible. We have tried to present the combinatorial ideas in an intuitive and geometric language, avoiding the often formal language or the language of computer programs common in the literature. Although there are some references to the literature (most in *Answers and References* at the end), there is no attempt at a survey.[1]

Although most of the material presented here is known in some form, there are a few novelties. One is our "matrix-ball" algorithm for the Robinson–Schensted–Knuth correspondence, which seems clearer and more efficient than others we have seen, and for which the basic symmetry property is evident. This generalizes an algorithm of Viennot for permutations; a similar algorithm has been developed independently by Stanley, Fomin, and Roby. Appendix A contains similar "matrix-ball" algorithms for variations of this correspondence. In addition, this appendix contains correspondences between skew tableaux, related to the Littlewood–Richardson rule, which generalize recent work of Haiman.

Our proof of the Littlewood–Richardson rule seems simpler than other published versions. The construction given in Chapter 8 of a "Schur" or "Weyl" module E^λ from a module E over a commutative ring, and a partition λ, should be better known than it is. Unlike most combinatorics texts, we have preferred to develop the whole theory for general Young tableaux, rather than just tableaux with distinct entries, even where results for the general case can be deduced from those for the special tableaux. In Appendix B we show how to construct the homology class of an algebraic variety, using only properties of singular homology and cohomology found in standard graduate topology texts.

In the combinatorial chapters, it is possible to keep the presentation essen-

[1] In this subject, where many ideas have been rediscovered or reinterpreted over and over, such a survey would be a formidable task. In the same spirit, the references to the literature are intended to point the reader to a place to learn about a topic; there is little attempt to assign credit or find original sources.

tially self-contained. The other chapters, however, are aimed at the relations of these combinatorial ideas to representation theory and geometry. Here the goal of self-containment works against that of building bridges, and we have frankly chosen the latter goal. Although we have tried to make these chapters accessible to those with only a rudimentary knowledge of representation theory, algebraic geometry, and topology, the success of some sections may depend on the reader's background, or at least on a willingness to accept some basic facts from these subjects.

For more about what is in the book, as well as the background assumed, see the section on notation, and the introductions to the three parts. A few words should perhaps be added about what is not in this book. For much of the story presented here there are analogues for general semisimple groups, or at least the other classical groups, with a continually growing literature. We have not discussed any of this here; however, we have tried to develop the subject for the general linear group so that a reader encountering the general case will have some preparation. We have not discussed other notions about tableaux, such as bitableaux or shifted tableaux, that are used primarily for representations of other groups. Although some constructions are made over the integers or arbitrary base rings – and we have preferred such constructions whenever possible – we do not enter into the distinctive features of positive characteristic. Finally, we have not presented the combinatorial algorithms in the format of computer programs; this was done, not to discourage the reader from writing such programs, but in the hope that a more intuitive and geometric discussion will make this subject attractive to those besides combinatorialists.

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William Fulton
February 1996

Dedicated to the memory of Birger Iversen.