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Cohen–Macaulay Rings

Revised edition

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For our wives,

Ulrike and Maja

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Preface to the revised edition

The main change in the revised edition is the new Chapter 10 on tight closure. This theory was created by Mel Hochster and Craig Huneke about ten years ago and is still strongly expanding. We treat the basic ideas, F-regular rings, and F-rational rings, including Smith's theorem by which F-rationality implies pseudo-rationality. Among the numerous applications of tight closure we have selected the Briançon–Skoda theorem and the theorem of Hochster and Huneke saying that equicharacteristic direct summands of regular rings are Cohen–Macaulay. To cover these applications, Section 8.4, which develops the technique of reduction to characteristic p, had to be rewritten. The title of Part III, no longer appropriate, has been changed.

Another noteworthy addition are the theorems of Gotzmann in the new Section 4.3. We believe that Chapter 4 now treats all the basic theorems on Hilbert functions. Moreover, this chapter has been slightly reorganized.

The new Section 5.5 contains a proof of Hochster's formula for the Betti numbers of a Stanley–Reisner ring since the free resolutions of such rings have recently received much attention. In the first edition the formula was used without proof.

We are grateful to all the readers of the first edition who have suggested corrections and improvements. Our special thanks go to L. Avramov, A. Conca, S. Iyengar, R. Y. Sharp, B. Ulrich, and K.-i. Watanabe.

Osnabrück and Essen,
October 1997

WINFRIED BRUNS
JÜRGEN HERZOG

Preface to the first edition

The notion of a Cohen–Macaulay ring marks the cross-roads of two powerful lines of research in present-day commutative algebra. While its main development belongs to the homological theory of commutative rings, it finds surprising and fruitful applications in the realm of algebraic combinatorics. Consequently this book is an introduction to the homological and combinatorial aspects of commutative algebra.

We have tried to keep the text self-contained. However, it has not proved possible, and would perhaps not have been appropriate, to develop commutative ring theory from scratch. Instead we assume the reader has acquired some fluency in the language of rings, ideals, and modules by working through an introductory text like Atiyah and Macdonald [15] or Sharp [344]. Nevertheless, to ease the access for the non-expert, the essentials of dimension theory have been collected in an appendix.

As exemplified by Matsumura's standard textbook [270], it is natural to have the notions of grade and depth follow dimension theory, and so Chapter 1 opens with the introduction of regular sequences on which their definition is based. From the very beginning we stress their connection with homological and linear algebra, and in particular with the Koszul complex.

Chapter 2 introduces Cohen–Macaulay rings and modules, our main subjects. Next we study regular local rings. They form the most special class of Cohen–Macaulay rings; their theory culminates in the Auslander–Buchsbaum–Serre and Auslander–Buchsbaum–Nagata theorems. Unlike the Cohen–Macaulay property in general, regularity has a very clear geometric interpretation: it is the algebraic counterpart of the notion of a non-singular point. Similarly the third class of rings introduced in Chapter 2, that of complete intersections, is of geometric significance.

In Chapter 3 a new homological aspect determines the development of the theory, namely the existence of injective resolutions. It leads us to the study of Gorenstein rings which in several respects are distinguished by their duality properties. When a Cohen–Macaulay local ring is not Gorenstein, then (almost always) it has at least a canonical module which, so to speak, acts as its natural partner in duality theorems, a decisive fact for many combinatorial applications. We then introduce local cohomology and prove Grothendieck's vanishing and local duality theorems.

Chapter 4 contains the combinatorial theory of commutative rings which mainly consists in the study of the Hilbert function of a graded module and the numerical invariants derived from it. A central point is Macaulay's theorem describing all possible Hilbert functions of homogeneous rings by a numerical condition. The intimate connection between homological and combinatorial data is displayed by several theorems, among them Stanley's characterization of Gorenstein domains. In the second part of this chapter the method of associated rings and modules is developed and used for assigning numerical invariants to modules over local rings.

Chapters 1–4 form the first part of the book. We consider this material as basic. The second part consists of Chapters 5–7 each of which is devoted to a special class of rings.

Chapter 5 contains the theory of Stanley–Reisner rings of simplicial complexes. Its main goal is the proof of Stanley's upper bound theorem for simplicial spheres. The transformation of this topological notion into an algebraic condition is through Hochster's theorem which relates simplicial homology and local cohomology. Furthermore we study the Gorenstein property for simplicial complexes and their canonical modules.

In Chapter 6 we investigate normal semigroup rings. The combinatorial object represented by a normal semigroup ring is the set of lattice points within a convex cone. According to a theorem of Hochster, normal semigroup rings are Cohen–Macaulay. Again the crucial point is the interplay between cellular homology on the geometric side and local cohomology on the algebraic. The fact that the ring of invariants of a linear torus action on a polynomial ring is a normal semigroup ring leads us naturally to the study of invariant rings, in particular those of finite groups. The chapter closes with the Hochster–Roberts theorem by which a ring of invariants of a linearly reductive group is Cohen–Macaulay.

Chapter 7 is devoted to determinantal rings. They are discussed in the framework of Hodge algebras and algebras with straightening laws. We establish the straightening laws of Hodge and of Doubilet, Rota, and Stein, prove that determinantal rings are Cohen–Macaulay, compute their canonical module, and determine the Gorenstein rings among them. In view of the extensive treatment available in [61], we have restricted this chapter to the absolutely essential.

The third part of the book is constituted by Chapters 8 and 9. They owe their existence to the fact that a Noetherian local ring is in general not Cohen–Macaulay. But Hochster has shown that such a ring possesses a (not necessarily finite) Cohen–Macaulay module, at least when it contains a field. The construction of these 'big' Cohen–Macaulay modules in Chapter 8 is a paradigm of characteristic p methods in commutative algebra, and we hope that it will prepare the reader for the more recent developments in this area which are centered around the notion of tight closure introduced by Hochster and Huneke [190].

In Chapter 9 we deduce the consequences of the existence of big Cohen–Macaulay modules, for example the intersection theorems of Peskine and Szpiro and Roberts, the Evans–Griffith syzygy theorem, and bounds for the Bass numbers of a module.

Chapters 8 and 9 are completely independent of Chapters 4–7, and the reader who is only interested in the homological theory may proceed from the end of Section 3.5 directly to Chapter 8.

It is only to be expected that the basic notions of homological algebra are ubiquitous in our book. But most of the time we will only use the long exact sequences for Ext and Tor, and the behaviour of these functors under flat extensions. Where we go beyond that, we have inserted a reference to Rotman [318]. One may regard it as paradoxical that we freely use the Ext functors while Chapter 3 contains a complete treatment of injective modules. However, their theory has several peculiar aspects so that we thought such a treatment would be welcomed by many readers.

The book contains numerous exercises. Some of them will be used in the main text. For these we have provided hints or even references to the literature, unless their solutions are completely straightforward. A reference of type A.n points to a result in the appendix.

Parts of this book were planned while we were guests of the Mathematisches Forschungsinstitut Oberwolfach. We thank the Forschungsinstitut for its generous hospitality.

We are grateful to all our friends, colleagues, and students, among them L. Avramov, C. Baeţica, M. Barile, A. Conca, H.–B. Foxby, C. Huneke, D. Popescu, P. Schenzel, and W. Vasconcelos who helped us by providing valuable information and by pointing out mistakes in preliminary versions. Our sincere thanks go to H. Matsumura and R. Sharp for their support in the early stages of this project.

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Vechta and Essen,
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