Part I

Basic concepts

1 Regular sequences and depth

After dimension, depth is the most fundamental numerical invariant of a Noetherian local ring R or a finite R-module M. While depth is defined in terms of regular sequences, it can be measured by the (non-)vanishing of certain Ext modules. This connection opens commutative algebra to the application of homological methods. Depth is connected with projective dimension and several notions of linear algebra over Noetherian rings.

Equally important is the description of depth (and its global relative grade) in terms of the Koszul complex which, in a sense, holds an intermediate position between arithmetic and homological algebra.

This introductory chapter also contains a section on graded rings and modules. These allow a decomposition of their elements into homogeneous components and therefore have a more accessible structure than rings and modules in general.

1.1 Regular sequences

Let M be a module over a ring R. We say that $x \in R$ is an *M-regular element* if $xz = 0$ for $z \in M$ implies $z = 0$, in other words, if x is not a zero-divisor on M. Regular sequences are composed of successively regular elements:

Definition 1.1.1. A sequence $\boldsymbol{x} = x_1, \dots, x_n$ of elements of R is called an *M-regular sequence* or simply an *M-sequence* if the following conditions are satisfied: (i) x_i is an $M/(x_1, \dots, x_{i-1})M$-regular element for $i = 1, \dots, n$, and (ii) $M/\boldsymbol{x}M \neq 0$.

In this situation we shall sometimes say that M is an *$\boldsymbol{x}$-regular module.* A *regular sequence* is an R-sequence.

A *weak M-sequence* is only required to satisfy condition (i).

Very often R will be a local ring with maximal ideal $\mathfrak{m}$, and $M \neq 0$ a finite R-module. If $\boldsymbol{x} \subset \mathfrak{m}$, then condition (ii) is satisfied automatically because of Nakayama's lemma.

The classical example of a regular sequence is the sequence $X_1, \dots, X_n$ of indeterminates in a polynomial ring $R = S[X_1, \dots, X_n]$. Conversely we shall see below that an M-sequence behaves to some extent like a sequence of indeterminates; this will be made precise in 1.1.8.

The next proposition contains a condition under which a regular sequence stays regular when the module or the ring is extended.

Proposition 1.1.2. *Let R be a ring, M an R-module, and $\boldsymbol{x} \subset R$ a weak M-sequence. Suppose $\varphi : R \to S$ is a ring homomorphism, and N an R-flat S-module. Then $\boldsymbol{x} \subset R$ and $\varphi(\boldsymbol{x}) \subset S$ are weak $(M \otimes_R N)$-sequences. If $\boldsymbol{x}(M \otimes_R N) \neq M \otimes_R N$, then $\boldsymbol{x}$ and $\varphi(\boldsymbol{x})$ are $(M \otimes_R N)$-sequences.*

PROOF. Multiplication by x_i is the same operation on $M \otimes N$ as multiplication by $\varphi(x_i)$; so it suffices to consider $\boldsymbol{x}$. The homothety $x_1 : M \to M$ is injective, and $x_1 \otimes N$ is injective too, because N is flat. Now $x_1 \otimes N$ is just multiplication by x_1 on $M \otimes N$. So x_1 is an $(M \otimes N)$-regular element. Next we have $(M \otimes N)/x_1(M \otimes N) \cong (M/x_1M) \otimes N$; an inductive argument will therefore complete the proof. □

The most important special cases of 1.1.2 are given in the following corollary. In its part (b) we use $\hat{M}$ to denote the $\mathfrak{m}$-adic completion of a module M over a local ring $(R, \mathfrak{m}, k)$ (by this notation we indicate that R has maximal ideal $\mathfrak{m}$ and residue class field $k = R/\mathfrak{m}$).

Corollary 1.1.3. *Let R be a Noetherian ring, M a finite R-module, and $\boldsymbol{x}$ an M-sequence.*
(a) *Suppose that a prime ideal $\mathfrak{p} \in \operatorname{Supp} M$ contains $\boldsymbol{x}$. Then $\boldsymbol{x}$ (as a sequence in $R_\mathfrak{p}$) is an $M_\mathfrak{p}$-sequence.*
(b) *Suppose that R is local with maximal ideal $\mathfrak{m}$. Then $\boldsymbol{x}$ (as a sequence in $\hat{R}$) is an $\hat{M}$-sequence.*

PROOF. Both the extensions $R \to R_\mathfrak{p}$ and $R \to \hat{R}$ are flat. (a) By hypothesis $M_\mathfrak{p} \neq 0$, and Nakayama's lemma implies $M_\mathfrak{p} \neq \mathfrak{p}M_\mathfrak{p}$. *A fortiori* we have $\boldsymbol{x}M_\mathfrak{p} \neq M_\mathfrak{p}$. (b) It suffices to note that $\hat{M} = M \otimes \hat{R}$ is a finite $\hat{R}$-module. □

The interplay between regular sequences and homological invariants is a major theme of this book, and numerous arguments will be based on the next proposition.

Proposition 1.1.4. *Let R be a ring, M an R-module, and $\boldsymbol{x}$ a weak M-sequence. Then an exact sequence*

$$N_2 \xrightarrow{\varphi_2} N_1 \xrightarrow{\varphi_1} N_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

of R-modules induces an exact sequence

$$N_2/\boldsymbol{x}N_2 \longrightarrow N_1/\boldsymbol{x}N_1 \longrightarrow N_0/\boldsymbol{x}N_0 \longrightarrow M/\boldsymbol{x}M \longrightarrow 0.$$

PROOF. By induction it is enough to consider the case in which $\boldsymbol{x}$ consists of a single M-regular element x. We obtain the induced sequence if we tensor the original one by $R/(x)$. Since tensor product is a right exact functor, we only need to verify exactness at N_1/xN_1. Let $^-$ denote residue classes modulo x. If $\bar{\varphi}_1(\bar{y}) = 0$, then $\varphi_1(y) = xz$ for some $z \in N_0$ and

$x\varphi_0(z) = 0$. By hypothesis we have $\varphi_0(z) = 0$; hence there is $y' \in N_1$ with $z = \varphi_1(y')$. It follows that $\varphi_1(y - xy') = 0$. So $y - xy' \in \varphi_2(N_2)$, and $\bar{y} \in \bar{\varphi}_2(\bar{N}_2)$ as desired. □

If we want to preserve the exactness of a longer sequence, then we need a stronger hypothesis.

Proposition 1.1.5. *Let R be a ring and*

$$N_\bullet: \cdots \longrightarrow N_m \xrightarrow{\varphi_m} N_{m-1} \longrightarrow \cdots \longrightarrow N_0 \xrightarrow{\varphi_0} N_{-1} \longrightarrow 0$$

an exact complex of R-modules. If $\boldsymbol{x}$ is weakly N_i-regular for all i, then $N_\bullet \otimes R/(\boldsymbol{x})$ is exact again.

PROOF. Once more one uses induction on the length of the sequence $\boldsymbol{x}$. So it is enough to treat the case $\boldsymbol{x} = x$. Since x is regular on N_i, it is regular on $\operatorname{Im}\varphi_{i+1}$ too. Therefore we can apply 1.1.4 to each exact sequence $N_{i+3} \to N_{i+2} \to N_{i+1} \to \operatorname{Im}\varphi_{i+1} \to 0$. □

Easy examples show that a permutation of a regular sequence need not be a regular sequence; see 1.1.13. Nevertheless there are natural conditions under which regular sequences can be permuted.

Let x_1, x_2 be an M-sequence, and denote the kernel of the multiplication by x_2 on M by K. Suppose that $z \in K$. Then we must have $z \in x_1M$, $z = x_1z'$, and $x_1(x_2z') = 0$, whence $x_2z' = 0$ and $z' \in K$, too. This shows $K = x_1K$ so that $K = 0$ if Nakayama's lemma is applicable. Somewhat surprisingly, x_1 is always regular on M/x_2M; the reader may check this easily.

Proposition 1.1.6. *Let R be a Noetherian local ring, M a finite R-module, and $\boldsymbol{x} = x_1, \dots, x_n$ an M-sequence. Then every permutation of $\boldsymbol{x}$ is an M-sequence.*

PROOF. Every permutation is a product of transpositions of adjacent elements. Therefore it is enough to show that $x_1, \dots, x_{i+1}, x_i, \dots, x_n$ is an M-sequence. The hypothesis of the proposition is satisfied for $\bar{M} = M/(x_1, \dots, x_{i-1})M$ and the $\bar{M}$-sequence $x_i, \dots, x_n$. So it suffices to treat the case $i = 1$ and to show that x_2, x_1 is an M-sequence. In view of the discussion above we only need to appeal to Nakayama's lemma. □

Quasi-regular sequences. Let R be a ring, M an R-module, and $\boldsymbol{X} = X_1, \dots, X_n$ be indeterminates over R. Then we write $M[\boldsymbol{X}]$ for $M \otimes R[\boldsymbol{X}]$ and call its elements *polynomials with coefficients in M*. If $\boldsymbol{x} = x_1, \dots, x_n$ is a sequence of elements of R, then the substitution $X_i \mapsto x_i$ induces an R-algebra homomorphism $R[\boldsymbol{X}] \to R$ and also an R-module homomorphism $M[\boldsymbol{X}] \to M$. We write $F(\boldsymbol{x})$ for the image of $F \in M[\boldsymbol{X}]$ under this map. (Since the monomials form a basis of the free R-module $R[\boldsymbol{X}]$, we may speak of the coefficients and the degree of an element of $M[\boldsymbol{X}]$.)

Theorem 1.1.7 (Rees). *Let R be a ring, M an R-module, $\boldsymbol{x} = x_1, \dots, x_n$ an M-sequence, and $I = (x_1, \dots, x_n)$. Let $\boldsymbol{X} = X_1, \dots, X_n$ be indeterminates over R. If $F \in M[\boldsymbol{X}]$ is homogeneous of (total) degree d and $F(\boldsymbol{x}) \in I^{d+1}M$, then the coefficients of F are in IM.*

Proof. We use induction on n. The case $n = 1$ is easy. Let $n > 1$ and suppose that the theorem holds for regular sequences of length at most $n-1$. We must first prove an auxiliary result which is an interesting fact in itself: *let $J = (x_1, \dots, x_{n-1})$; then x_n is regular on M/J^jM for all $j \geq 1$.*

In fact, suppose that $x_n y \in J^jM$ for some $j > 1$. Arguing by induction we have $y \in J^{j-1}M$; so $y = G(x_1, \dots, x_{n-1})$ where $G \in M[X_1, \dots, X_{n-1}]$ is homogeneous of degree $j-1$. Set $G' = x_n G$. Then the theorem applied to $G' \in M[X_1, \dots, X_{n-1}]$ yields that the coefficients of G' are in JM. Since x_n is regular modulo JM, it follows that the coefficients of G are in JM too, and therefore $y \in J^jM$.

The proof of the theorem for sequences of length n requires induction on d. The case $d = 0$ is trivial. Assume that $d > 0$. First we reduce to the case in which $F(\boldsymbol{x}) = 0$. Since $F(\boldsymbol{x}) \in I^{d+1}M$, one has $F(\boldsymbol{x}) = G(\boldsymbol{x})$ with G homogeneous of degree $d+1$. Then $G = \sum_{i=1}^n X_i G_i$ with G_i homogeneous of degree d. Set $G'_i = x_i G_i$ and $G' = \sum_{i=1}^n G'_i$. So $F - G'$ is homogeneous of degree d, and $(F - G')(\boldsymbol{x}) = 0$. Furthermore, $F - G'$ has coefficients in IM if and only if this holds for F.

Thus assume that $F(\boldsymbol{x}) = 0$. Then we write $F = G + X_n H$ with $G \in M[X_1, \dots, X_{n-1}]$. The auxiliary claim above implies that $H(\boldsymbol{x}) \in J^dM \subset I^dM$. By induction on d the coefficients of H are in IM. On the other hand $H(\boldsymbol{x}) = H'(x_1, \dots, x_{n-1})$ with $H' \in M[X_1, \dots, X_{n-1}]$ homogeneous of degree d. As

$$(G + x_n H')(x_1, \dots, x_{n-1}) = F(\boldsymbol{x}) = 0,$$

it follows by induction on n that $G + x_n H'$ has coefficients in JM. Since $x_n H'$ has its coefficients in IM, the coefficients of G must be in IM too. □

Let I be an ideal in R. One defines the *associated graded ring of R with respect to I* by

$$\operatorname{gr}_I(R) = \bigoplus_{i=0}^{\infty} I^i/I^{i+1}.$$

The multiplication in $\operatorname{gr}_I(R)$ is induced by the multiplication $I^i \times I^j \to I^{i+j}$, and $\operatorname{gr}_I(R)$ is a graded ring with $(\operatorname{gr}_I(R))_0 = R/I$. If M is an R-module, one similarly constructs the *associated graded module*

$$\operatorname{gr}_I(M) = \bigoplus_{i=0}^{\infty} I^iM/I^{i+1}M.$$

It is straightforward to verify that $\mathrm{gr}_I(M)$ is a graded $\mathrm{gr}_I(R)$-module. (Graded rings and modules will be discussed in Section 1.5. The reader not familiar with the basic terminology may wish to consult 1.5.) Let I be generated by $x_1, \dots, x_n$. Then one has a natural surjection $R[X] = R[X_1, \dots, X_n] \to \mathrm{gr}_I(R)$ which is induced by the natural homomorphism $R \to R/I$ and the substitution $X_i \mapsto \bar{x}_i \in I/I^2$. Similarly there is an epimorphism $\psi : M[X] \to \mathrm{gr}_I(M)$. One first defines ψ on the homogeneous components by assigning to a homogeneous polynomial $F \in M[X]$ of degree d the residue class of $F(\boldsymbol{x})$ in $I^d M/I^{d+1}M$; then ψ is extended additively. As the reader may check, ψ is an epimorphism of graded $R[X]$-modules. Obviously $IM[X] \subset \operatorname{Ker}\psi$; via the identification $M[X]/IM[X] \cong (M/IM)[X]$, we therefore get an induced epimorphism $\varphi : (M/IM)[X] \to \mathrm{gr}_I(M)$. The kernel of ψ is generated by the homogeneous polynomials $F \in M[X]$ of degree d, $d \in \mathbb{N}$, such that $F(\boldsymbol{x}) \in I^{d+1}M$. So we obtain as a reformulation of 1.1.7

Theorem 1.1.8. *Let R be a ring, M an R-module, $\boldsymbol{x} = x_1, \dots, x_n$ an M-sequence, and $I = (\boldsymbol{x})$. Then the map $(M/IM)[X_1, \dots, X_n] \to \mathrm{gr}_I(M)$ induced by the substitution $X_i \mapsto \bar{x}_i \in I/I^2$ is an isomorphism.*

This theorem says very precisely to what extent a regular sequence resembles a sequence of indeterminates: the residue classes $\bar{x}_i \in I/I^2$ operate on $\mathrm{gr}_I(M)$ exactly like indeterminates. Since a regular sequence may lose regularity under a permutation, whereas 1.1.8 is independent of the order in which $\boldsymbol{x}$ is given, it is not possible to reverse 1.1.8; see however 1.1.15. Later on it will be useful to have a name for sequences $\boldsymbol{x}$ satisfying the conclusion of 1.1.8; we call them *M-quasi-regular* if, in addition, $\boldsymbol{x}M \neq M$.

Exercises

1.1.9. Let $0 \to U \to M \to N \to 0$ be an exact sequence of R-modules, and $\boldsymbol{x}$ a sequence which is weakly U-regular and (weakly) N-regular. Prove that $\boldsymbol{x}$ is (weakly) M-regular too.

1.1.10. (a) Let $x_1, \dots, x_i, \dots, x_n$ and $x_1, \dots, x_i', \dots, x_n$ be (weakly) M-regular. Show that $x_1, \dots, x_i x_i', \dots, x_n$ is (weakly) M-regular. (Hint: In the essential case $i = 1$ one finds an exact sequence as in 1.1.9 with $M/x_1x_1'M$ as the middle term.)
(b) Prove that $x_1^{e_1}, \dots, x_n^{e_n}$ is (weakly) M-regular for all $e_i \geq 1$.

1.1.11. Prove that the converse of 1.1.2 holds if, in the situation of 1.1.2, N is faithfully flat over R.

1.1.12. (a) Prove that if $\boldsymbol{x}$ is a weak M-sequence, then $\mathrm{Tor}_1^R(M, R/(\boldsymbol{x})) = 0$.
(b) Prove that if, in addition, $\boldsymbol{x}$ is a weak R-sequence, then $\mathrm{Tor}_i^R(M, R/(\boldsymbol{x})) = 0$ for all $i \geq 1$.

1.1.13. Let $R = K[X, Y, Z]$, k a field. Show that X, $Y(1 - X)$, $Z(1 - X)$ is an R-sequence, but $Y(1 - X)$, $Z(1 - X)$, X is not.

1.1.14. Prove that $x_1,\dots,x_n$ is M-quasi-regular if and only if $\bar{x}_1,\dots,\bar{x}_n \in I/I^2$ is a $\mathrm{gr}_I(M)$-regular sequence where $I = (x_1,\dots,x_n)$.

1.1.15. Suppose that $\boldsymbol{x}$ is M-quasi-regular, and let $I = (x_1,\dots,x_n)$. Prove
(a) if $x_1 z \in I^i M$ for $z \in M$, then $z \in I^{i-1}M$,
(b) $x_2,\dots,x_n$ is (M/x_1M)-quasi-regular,
(c) if R is Noetherian local and M is finite, then $\boldsymbol{x}$ is an M-sequence.

1.2 Grade and depth

Let R be a Noetherian ring and M an R-module. If $\boldsymbol{x} = x_1,\dots,x_n$ is an M-sequence, then the sequence $(x_1) \subset (x_1,x_2) \subset \cdots \subset (x_1,\dots,x_n)$ ascends strictly for obvious reasons. Therefore an M-sequence can be extended to a maximal such sequence: an M-sequence $\boldsymbol{x}$ (contained in an ideal I) is *maximal* (in I), if $x_1,\dots,x_{n+1}$ is not an M-sequence for any $x_{n+1} \in R$ ($x_{n+1} \in I$). We will prove that all maximal M-sequences in an ideal I with $IM \neq M$ have the same length if M is finite. This allows us to introduce the fundamental notions of grade and depth.

In connection with regular sequences, finite modules over Noetherian rings are distinguished for two reasons: first, every zero-divisor of M is contained in an associated prime ideal, and, second, the number of these prime ideals is finite. Both facts together imply the following proposition that is 'among the most useful in the theory of commutative rings' (Kaplansky [231], p. 56).

Proposition 1.2.1. *Let R be a Noetherian ring, and M a finite R-module. If an ideal $I \subset R$ consists of zero-divisors of M, then $I \subset \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass} M$.*

PROOF. If $I \not\subset \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass} M$, then there exists $a \in I$ with $a \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass} M$. This follows immediately from 1.2.2. □

The following lemma, which we have just used in its simplest form, is the standard argument of 'prime avoidance'.

Lemma 1.2.2. *Let R be a ring, $\mathfrak{p}_1,\dots,\mathfrak{p}_m$ prime ideals, M an R-module, and $x_1,\dots,x_n \in M$. Set $N = \sum_{i=1}^n Rx_i$. If $N_{\mathfrak{p}_j} \not\subset \mathfrak{p}_j M_{\mathfrak{p}_j}$ for $j = 1,\dots,m$, then there exist $a_2,\dots,a_n \in R$ such that $x_1 + \sum_{i=2}^n a_i x_i \notin \mathfrak{p}_j M_{\mathfrak{p}_j}$ for $j = 1,\dots,m$.*

PROOF. We use induction on m, and so suppose that there are $a'_2,\dots,a'_n \in R$ for which $x'_1 = x_1 + \sum_{i=2}^n a'_i x_i \notin \mathfrak{p}_j M_{\mathfrak{p}_j}$ for $j = 1,\dots,m-1$. Moreover, it is no restriction to assume that the $\mathfrak{p}_i$ are pairwise distinct and that $\mathfrak{p}_m$ is a minimal member of $\mathfrak{p}_1,\dots,\mathfrak{p}_m$. So there exists $r \in (\bigcap_{j=1}^{m-1} \mathfrak{p}_j) \setminus \mathfrak{p}_m$. Put $x'_i = r x_i$ for $i = 2,\dots,n$ and $N' = \sum_{i=1}^n Rx'_i$. Since $r \notin \mathfrak{p}_m$ we have $N'_{\mathfrak{p}_m} = N_{\mathfrak{p}_m}$. On the other hand, as $r \in \mathfrak{p}_j$ for $j = 1,\dots,m-1$, it follows that $x'_1 + x'_i \notin \mathfrak{p}_j M_{\mathfrak{p}_j}$ for $i = 2,\dots,n$ and $j = 1,\dots,m-1$. If $x'_1 \notin \mathfrak{p}_m M_{\mathfrak{p}_m}$, then x'_1 is the element desired; otherwise $x'_1 + x'_i \notin \mathfrak{p}_m M_{\mathfrak{p}_m}$ for some $i \in \{2,\dots,n\}$, and we choose $x'_1 + x'_i$. □

Note that if $M = R$ and $N = I \subset R$, then the condition $N_{\mathfrak{p}_j} \not\subset \mathfrak{p}_j M_{\mathfrak{p}_j}$ simplifies to $I \not\subset \mathfrak{p}_j$.

Suppose that an ideal I is contained in $\mathfrak{p} \in \operatorname{Ass} M$. By definition, there exists $z \in M$ with $\mathfrak{p} = \operatorname{Ann} z$. Hence the assignment $1 \mapsto z$ induces a monomorphism $\varphi' : R/\mathfrak{p} \to M$, and thus a non-zero homomorphism $\varphi : R/I \to M$. This simple observation allows us to describe in homological terms that a certain ideal consists of zero-divisors:

Proposition 1.2.3. *Let R be a ring, and M, N R-modules. Set $I = \operatorname{Ann} N$.*
(a) *If I contains an M-regular element, then* $\operatorname{Hom}_R(N, M) = 0$.
(b) *Conversely, if R is Noetherian, and M, N are finite,* $\operatorname{Hom}_R(N, M) = 0$ *implies that I contains an M-regular element.*

PROOF. (a) is evident. (b) Assume that I consists of zero-divisors of M, and apply 1.2.1 to find a $\mathfrak{p} \in \operatorname{Ass} M$ such that $I \subset \mathfrak{p}$. By hypothesis, $\mathfrak{p} \in \operatorname{Supp} N$; so $N_\mathfrak{p} \otimes k(\mathfrak{p}) \neq 0$ by Nakayama's lemma, and since $N_\mathfrak{p} \otimes k(\mathfrak{p})$ is just a direct sum of copies of $k(\mathfrak{p})$, one has an epimorphism $N_\mathfrak{p} \to k(\mathfrak{p})$. (By $k(\mathfrak{p})$ we denote the residue class field $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ of $R_\mathfrak{p}$.) Note that $\mathfrak{p}R_\mathfrak{p} \in \operatorname{Ass} M_\mathfrak{p}$. Hence the observation above yields a non-zero $\varphi' \in \operatorname{Hom}_{R_\mathfrak{p}}(N_\mathfrak{p}, M_\mathfrak{p})$. Since $\operatorname{Hom}_{R_\mathfrak{p}}(N_\mathfrak{p}, M_\mathfrak{p}) \cong \operatorname{Hom}_R(N, M)_\mathfrak{p}$, it follows that $\operatorname{Hom}_R(N, M) \neq 0$. (See [318], Theorem 3.84 for the isomorphism just applied.) $\square$

Lemma 1.2.4. *Let R be a ring, M, N be R-modules, and $\boldsymbol{x} = x_1, \dots, x_n$ a weak M-sequence in* $\operatorname{Ann} N$. *Then*

$$\operatorname{Hom}_R(N, M/\boldsymbol{x}M) \cong \operatorname{Ext}_R^n(N, M).$$

PROOF. We use induction on n, starting from the vacuous case $n = 0$. Let $n \geq 1$, and set $\boldsymbol{x}' = x_1, \dots, x_{n-1}$. Then the induction hypothesis implies that $\operatorname{Ext}_R^{n-1}(N, M) \cong \operatorname{Hom}_R(N, M/\boldsymbol{x}'M)$. As x_n is $(M/\boldsymbol{x}'M)$-regular, $\operatorname{Ext}_R^{n-1}(N, M) = 0$ by 1.2.3. Therefore the exact sequence

$$0 \longrightarrow M \xrightarrow{x_1} M \longrightarrow M/x_1 M \longrightarrow 0$$

yields an exact sequence

$$0 \longrightarrow \operatorname{Ext}_R^{n-1}(N, M/xM) \xrightarrow{\psi} \operatorname{Ext}_R^n(N, M) \xrightarrow{\varphi} \operatorname{Ext}_R^n(N, M).$$

The map φ is multiplication by x_1 inherited from M, but multiplication by x_1 on N also induces φ; see [318], Theorem 7.16. Since $x_1 \in \operatorname{Ann} N$, one has $\varphi = 0$. Hence ψ is an isomorphism, and a second application of the induction hypothesis yields the assertion. $\square$

Let R be Noetherian, I an ideal, M a finite R-module with $M \neq IM$, and $\boldsymbol{x} = x_1, \dots, x_n$ a maximal M-sequence in I. From 1.2.3 and 1.2.4

we have, since I contains an $(M/(x_1,\dots,x_{i-1})M)$-regular element for $i=1,\dots,n$,

$$\operatorname{Ext}_R^{i-1}(R/I,M)\cong \operatorname{Hom}_R\big(R/I,M/(x_1,\dots,x_{i-1})M\big)=0.$$

On the other hand, since $IM\neq M$ and $\boldsymbol{x}$ is a maximal M-sequence in I, then I must consist of zero-divisors of $M/\boldsymbol{x}M$, whence

$$\operatorname{Ext}_R^n(R/I,M)\cong \operatorname{Hom}_R(R/I,M/\boldsymbol{x}M)\neq 0.$$

We have therefore proved

Theorem 1.2.5 (Rees). *Let R be a Noetherian ring, M a finite R-module, and I an ideal such that $IM\neq M$. Then all maximal M-sequences in I have the same length n given by*

$$n=\min\{i:\ \operatorname{Ext}_R^i(R/I,M)\neq 0\}.$$

Definition 1.2.6. Let R be a Noetherian ring, M a finite R-module, and I an ideal such that $IM\neq M$. Then the common length of the maximal M-sequences in I is called the *grade of I on M*, denoted by

$$\operatorname{grade}(I,M).$$

We complement this definition by setting $\operatorname{grade}(I,M)=\infty$ if $IM=M$. This is consistent with 1.2.5:

$$\operatorname{grade}(I,M)=\infty \quad\iff\quad \operatorname{Ext}_R^i(R/I,M)=0 \text{ for all } i.$$

For, if $IM=M$, then $\operatorname{Supp}M\cap\operatorname{Supp}R/I=\emptyset$ by Nakayama's lemma, hence

$$(1)\qquad \operatorname{Supp}\operatorname{Ext}_R^i(R/I,M)\subset \operatorname{Supp}M\cap\operatorname{Supp}R/I=\emptyset;$$

conversely, if $\operatorname{Ext}_R^i(R/I,M)=0$ for all i, then 1.2.5 gives $IM=M$.

The inclusion in (1) results from the natural isomorphism

$$\operatorname{Ext}_{R_\mathfrak{p}}^i(N_\mathfrak{p},M_\mathfrak{p})\cong \operatorname{Ext}_R^i(N,M)_\mathfrak{p}$$

which holds if R is Noetherian, N a finite R-module, M an arbitrary R-module, and $\mathfrak{p}\in\operatorname{Spec}R$; see [318], Theorem 9.50.

A special situation will occur so often that it merits a special notation:

Definition 1.2.7. Let $(R,\mathfrak{m},k)$ be a Noetherian local ring, and M a finite R-module. Then the grade of $\mathfrak{m}$ on M is called the *depth of M*, denoted

$$\operatorname{depth}M.$$

Because of its importance we repeat the most often used special case of 1.2.5:

Theorem 1.2.8. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, and M a finite non-zero R-module. Then* $\operatorname{depth} M = \min\{i : \operatorname{Ext}^i_R(k, M) \neq 0\}$.

Some formulas for grade. We now study the behaviour of $\operatorname{grade}(I, M)$ along exact sequences.

Proposition 1.2.9. *Let R be a Noetherian ring, $I \subset R$ an ideal, and $0 \to U \to M \to N \to 0$ an exact sequence of finite R-modules. Then*

$$\begin{aligned} \operatorname{grade}(I, M) &\geq \min\{\operatorname{grade}(I, U), \operatorname{grade}(I, N)\}, \\ \operatorname{grade}(I, U) &\geq \min\{\operatorname{grade}(I, M), \operatorname{grade}(I, N) + 1\}, \\ \operatorname{grade}(I, N) &\geq \min\{\operatorname{grade}(I, U) - 1, \operatorname{grade}(I, M)\}. \end{aligned}$$

PROOF. The given exact sequence induces a long exact sequence

$$\cdots \to \operatorname{Ext}^{i-1}_R(R/I, N) \to \operatorname{Ext}^i_R(R/I, U) \to \operatorname{Ext}^i_R(R/I, M) \\ \to \operatorname{Ext}^i_R(R/I, N) \to \operatorname{Ext}^{i+1}_R(R/I, U) \to \cdots$$

One observes that $\operatorname{Ext}^i_R(R/I, M) = 0$ if $\operatorname{Ext}^i_R(R/I, U)$ and $\operatorname{Ext}^i_R(R/I, N)$ both vanish. Therefore the first inequality follows from 1.2.5 and our discussion of the case $\operatorname{grade}(I, \_) = \infty$. Completely analogous arguments show the second and the third inequality. □

The next proposition collects some formulas which are useful in the computation of grades. (In the sequel $V(I)$ denotes the set of prime ideals containing I.)

Proposition 1.2.10. *Let R be a Noetherian ring, I, J ideals of R, and M a finite R-module. Then*

(a) $\operatorname{grade}(I, M) = \inf\{\operatorname{depth} M_\mathfrak{p} : \mathfrak{p} \in V(I)\}$,

(b) $\operatorname{grade}(I, M) = \operatorname{grade}(\operatorname{Rad} I, M)$,

(c) $\operatorname{grade}(I \cap J, M) = \min\{\operatorname{grade}(I, M), \operatorname{grade}(J, M)\}$,

(d) *if $\boldsymbol{x} = x_1, \ldots, x_n$ is an M-sequence in I, then* $\operatorname{grade}(I/(\boldsymbol{x}), M/\boldsymbol{x}M) = \operatorname{grade}(I, M/\boldsymbol{x}M) = \operatorname{grade}(I, M) - n$,

(e) *if N is a finite R-module with* $\operatorname{Supp} N = V(I)$, *then*

$$\operatorname{grade}(I, M) = \inf\{i : \operatorname{Ext}^i_R(N, M) \neq 0\}.$$

PROOF. (a) It is evident from the definition that $\operatorname{grade}(I, M) \leq \operatorname{grade}(\mathfrak{p}, M)$ for $\mathfrak{p} \in V(I)$, and it follows from 1.1.3 that $\operatorname{grade}(\mathfrak{p}, M) \leq \operatorname{depth} M_\mathfrak{p}$. Furthermore, if $\operatorname{grade}(I, M) = \infty$, then $\operatorname{Supp} M \cap V(I) = \emptyset$ so that $\operatorname{depth} M_\mathfrak{p} = \infty$ for all $\mathfrak{p} \in V(I)$. Thus suppose $IM \neq M$ and choose a maximal M-sequence $\boldsymbol{x}$ in I. By 1.2.1 there exists $\mathfrak{p} \in \operatorname{Ass}(M/\boldsymbol{x}M)$ with $I \subset \mathfrak{p}$. Since $\mathfrak{p}R_\mathfrak{p} \in \operatorname{Ass}(M/\boldsymbol{x}M)_\mathfrak{p}$ and $(M/\boldsymbol{x}M)_\mathfrak{p} \cong M_\mathfrak{p}/\boldsymbol{x}M_\mathfrak{p}$, the ideal