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1
Order-of-Magnitude Astrophysics

1.1 Introduction

The subject of astrophysics involves the application of the laws of physics to large macroscopic systems in order to understand their behaviour and predict new phenomena. This approach is similar in spirit to the application of the laws of physics in the study of, say, condensed-matter phenomena, except for the following three significant differences:

(1) We have far less control over the external conditions and parameters in astrophysics than in, say, condensed-matter physics. It is not possible to study systems under controlled conditions so that certain physical processes dominate the behaviour. Identifying the causes of various observed phenomena in astrophysics will require far greater reliance on statistical arguments than in laboratory physics.

(2) The astrophysical systems of interest span a wide range of parameter space and require inputs from several different branches of physics. Typically, the densities can vary from 10^{-25} gm cm^{-3} (interstellar medium) to 10^{15} gm cm^{-3} (neutron stars); temperatures from 2.7 K (microwave background radiation) to 10^9 K (accreting x-ray sources) or even to 10^{15} K (early universe); radiation from wavelengths of meters (radio waves) to fractions of angstroms (hard gamma rays); typical speeds of particles can go up to $0.99c$ (relativistic jets). Clearly we require inputs from quantum-mechanical and relativistic regimes as well as from more familiar classical physics.

(3) The primary source of information about distant astrophysical sources is the electromagnetic radiation detected from them. Therefore, to obtain a complete picture about any source, it is necessary to examine it in all the wave bands. Because of technological limitations, it is often quite difficult to have uniform coverage across the entire electromagnetic spectrum. Hence the information we have about the sources is often distorted or incomplete.

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These considerations suggest that two aspects will be most important in the study of different astrophysical systems. The first is the appreciation of the different states in which bulk matter can exist under different conditions and the dynamics of the matter governed by different equations of state. The second is an understanding of different radiative processes that lead to the emission of photons, which act as prime carriers of information about astronomical objects.

We shall be concerned with these and related topics in several chapters of this book. The purpose of this introductory chapter is twofold: It will first provide – in Sections (1.2) – (1.4) – a rapid overview of several physical processes at an order-of-magnitude level and introduce the necessary concepts. Then we will make an attempt to understand the existence of different astrophysical structures from first principles to the extent possible. Implementing such a plan, of course, has not been possible even in laboratory physics, and it is unlikely to succeed in the case of astrophysics. At present, astrophysics does require a fair amount of observational and phenomenological input, just like any other branch of applied physics. Nevertheless, we will make such an attempt as it is useful in providing the most basic and direct connection between physics and astrophysics.

1.2 Energy Scales of Physical Phenomena

Let us consider a system of N particles ($N \gg 1$), each of mass m, occupying a spherical region of radius R. In dealing with the dynamics of such a large collection of particles, it is useful to introduce the concept of pressure exerted by the system of particles as the momentum transferred per second normal to a (fictitious) surface of unit area. The contribution to the rate of momentum transfer (per unit area) from particles of energy ϵ is $n(\epsilon)\mathbf{p}(\epsilon) \cdot \mathbf{v}(\epsilon)$, where $n(\epsilon)$ denotes the number of particles per unit volume with momentum $\mathbf{p}(\epsilon)$ and velocity $\mathbf{v}(\epsilon)$. We obtain the net pressure by averaging this expression over the angles defined by $\mathbf{p}$ (or $\mathbf{v}$) and summing over all values of the energy. Because the momentum and the velocity are parallel to each other, the vector dot product $\mathbf{p} \cdot \mathbf{v}$ averages to $(1/3)pv$ (in three dimensions), giving

$$P = \frac{1}{3}\int_0^\infty n(\epsilon)p(\epsilon)v(\epsilon)\,d\epsilon, \tag{1.1}$$

where the integration is over all energies. The system is called ideal if the kinetic energy dominates over the interaction energy of the particles. In that case ϵ is essentially the kinetic energy of the particle. With the relations

$$p = \gamma m v, \quad \epsilon = (\gamma - 1)mc^2, \quad \gamma \equiv \left(1 - \frac{v^2}{c^2}\right)^{-1/2}, \tag{1.2}$$

where ϵ is the kinetic energy of the particle, the pressure can be expressed in the

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$$P = \frac{1}{3}\int_0^\infty n\epsilon\left(1+\frac{2mc^2}{\epsilon}\right)\left(1+\frac{mc^2}{\epsilon}\right)^{-1} d\epsilon. \tag{1.3}$$

In the nonrelativistic (NR) limit (with $mc^2 \gg \epsilon$), this gives $P_{\rm NR} \approx (2/3)\langle n\epsilon\rangle = (2/3)U_{\rm NR}$, where $U_{\rm NR}$ is the energy density (i.e., energy per unit volume) of the particles. In the relativistic case (with $\epsilon \gg mc^2$ or when the particles are massless), the corresponding expression is $P_{\rm ER} \approx (1/3)\langle n\epsilon\rangle = (1/3)U_{\rm NR}$. Hence, in general, $P \approx U$ up to a factor of order unity.

This result can be converted into a more useful form of equation of state whenever the mean free path of the particles in the system is small compared with the length scales over which the physical parameters of the system change significantly. Then the pressure can be expressed in terms of density and temperature if the energy density can be expressed in terms of these variables. This is possible in several contexts leading to different equations of state. To understand each of these cases it is useful to start by identifying the characteristic energy scales of bulk matter. We now turn to this task.

1.2.1 Rest-Mass Energy

We can associate the rest-mass energy mc^2 with each individual particle of mass m. In normal matter, made up of nucleons and electrons, the lowest value for rest-mass is provided by electrons with $m_ec^2 \approx 0.5$ MeV. For nucleons, the rest-mass energy is $m_pc^2 \approx 1$ GeV. Because the total mass of the system is mostly due to the nucleons, the total rest-mass energy will be $E_{\rm mass} \cong NAm_pc^2 \cong Mc^2$, where $Am_p \simeq m$ is the mass of each nucleus and $Nm = M$ is the total mass of the system. Rest-mass energy is extensive – that is, $E_{\rm mass} \propto N$ – in the low-energy phenomena in which masses of individual nuclei do not change.

1.2.2 Atomic Binding Energies

If the particles of the system have internal structure (molecular, atomic, nuclear, etc.) then we get further energy scales that are characteristic of the interactions. The simplest is the atomic binding energy of atoms and molecules, which arises from the electromagnetic coupling between the particles.

The Hamiltonian describing an electron, moving in the Coulomb field of a nucleus of charge Zq, is given by $H_0 = (p^2/2m_e) - (Zq^2/r)$. If this electron is described by a wave function $\psi(\mathbf{x}, L)$, where L denotes the characteristic scale over which ψ varies significantly, then the expectation value for the energy of the electron in this state is $E(L) = \langle\psi|H_0|\psi\rangle \approx (\hbar^2/2m_eL^2) - (Zq^2/L)$. The first term arises from the fact that $\langle\psi|p^2|\psi\rangle = -\hbar^2\langle\psi|\nabla^2|\psi\rangle \approx (\hbar^2/L^2)$, which is equivalent to the uncertainty principle stated in the form $p \cong \hbar/L$. This

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expression for $E(L)$ reaches a minimum value of $E_{\min} = -Z^2\epsilon_a$ when L is varied, with the minimum occurring at $L_{\min} = (a_0/Z)$, where

$$a_0 \equiv \frac{\hbar^2}{m_e q^2} \equiv \frac{\lambdabar_e}{\alpha} \approx 0.52 \times 10^{-8} \text{ cm}, \quad \epsilon_a \equiv \frac{m_e q^4}{2\hbar^2} = \frac{1}{2}\alpha^2 m_e c^2 \approx 13.6 \text{ eV}, \tag{1.4}$$

with the definitions $\lambdabar_e \equiv (\hbar/m_e c)$ and $\alpha \equiv (q^2/\hbar c)$. a_0 and ϵ_a correspond to the size and the ground-state energy of a hydrogen atom with $Z = 1$. The wavelength λ corresponding to ϵ_a is $\lambda = (hc/\epsilon_a) = 2\alpha^{-2}\lambdabar_e \simeq 10^3$ Å and lies in the UV band. The fine-structure constant $\alpha \approx 7.3 \times 10^{-3}$ plays an important role in the structure of matter and arises as the ratio between several interesting variables:

$$\alpha = (v/Zc) = (2\mu_B/qa_0) = (r_0/\lambdabar_e) = (\lambdabar_e/a_0),$$

where v is the speed of an electron in the atom, $\mu_B \equiv (q\hbar/2m_e c)$ is the Bohr magneton representing the magnetic moment of the electron, and $r_0 \equiv (q^2/m_e c^2)$ is called the classical electron radius.

When atoms of size a_0 are closely packed, the number density of atoms is $n_{\text{solid}} \approx (2a_0)^{-3} \approx 10^{24}$ cm^{-3}. The binding energy of such a solid arises essentially because of the residual electromagnetic force between the atoms, and the typical binding energy per particle is $f\epsilon_a$ with $f \approx (0.1-1)$.

1.2.3 Molecular Binding Energy

The simplest molecular structure consists of two atoms bound to each other in the form of a diatomic molecule. The effective potential energy of interaction $U(r)$ between the atoms in such a molecule arises from a residual electrostatic coupling and has a minimum at a separation $r \simeq a_0$, approximately the size of the atom. The depth of the potential well at the minimum is comparable with the electronic-energy level ϵ_a of the atom. In addition to the internal, electronic, binding energies of the atoms comprising the molecule, there are two other contributions to the energy of a diatomic molecule:

(1) The atoms of such a molecule can vibrate at some characteristic frequency ω_{vib} about the mean position along the line connecting them; this will lead to vibrational-energy levels separated by $E_{\text{vib}} \approx \hbar\omega_{\text{vib}}$. If the displacement is $\sim a_0$ from the minimum, the vibrational energy E_{vib} will be $\sim\epsilon_a$. Writing $\epsilon_a \approx (1/2)\mu\omega_{\text{vib}}^2 a_0^2 \cong (\hbar^2/m_e a_0^2)$, where μ is the reduced mass of the two atoms, we get

$$E_{\text{vib}} = \hbar\omega_{\text{vib}} \approx \frac{\hbar^2}{(\mu m_e)^{1/2} a_0^2} \approx \left(\frac{m_e}{\mu}\right)^{1/2} \epsilon_a \simeq 0.25 \text{ eV} \tag{1.5}$$

if $\mu \simeq m_p$.

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(2) The molecule can also rotate about an axis perpendicular to the line joining them. If the rotational angular momentum is J, this will contribute an energy of approximately

$$E_{\rm rot} \approx \left(\frac{J^2}{\mu a_0^2}\right) \approx \left(\frac{\hbar^2}{\mu a_0^2}\right) \approx \left(\frac{m_e}{\mu}\right)\epsilon_a \approx 10^{-3}\epsilon_a \simeq 10^{-2}\ \text{eV} \qquad (1.6)$$

if $J \simeq \hbar$ and $\mu \simeq m_p$. It follows from these relations that $E_{\rm rot}{:}E_{\rm vib}{:}E_0 \approx (m_e/\mu){:}(m_e/\mu)^{1/2}{:}1$ and $E_{\rm vib} \approx \sqrt{\epsilon_a E_{\rm rot}}$. Because $(m_e/\mu) \approx 10^{-3}$, the wavelengths of radiation from vibrational transitions are $\sim$40 times larger than those of electronic transitions; similarly, the rotational transitions lead to radiation with wavelengths $\sim$1000 times larger than those of electronic transitions. These wavelengths are usually in the IR band.

Atomic and molecular energies are also extensive, with the binding energy of a system of N particles scaling as N.

1.2.4 Nuclear-Energy Scales

Atomic nuclei are bound by the strong-interaction force that provides a binding energy per particle of $\sim$8 MeV, which is the characteristic scale for nuclear-energy levels. In the astrophysical context, a more relevant energy scale is the one at which nuclear reactions can be triggered in bulk matter, which can be estimated as follows. For two protons to fuse together while undergoing nuclear reaction, it is necessary that they be brought within the range of attractive nuclear force, which is approximately $l \approx (h/m_p c) = (2\pi\hbar/m_p c)$. Because this requires overcoming the Coloumb repulsion, such direct interaction can take place only if the kinetic energy of colliding particles is of the order of the electrostatic potential energy at the separation l. This requires energies of the order of $\epsilon \approx (q^2/l) = (\alpha/2\pi)m_p c^2 \approx 1$ MeV. It is, however, possible for nuclear reactions to occur through quantum-mechanical tunneling when the de Broglie wavelength $\lambda_{\rm deB} \equiv (h/m_p v) = l(c/v)$ of the two protons overlap. This occurs when the energy of the protons is approximately $\epsilon_{\rm nucl} \approx (\alpha^2/2\pi^2)m_p c^2 \approx 1$ keV. It is conventional to write this expression as $\epsilon_{\rm nucl} \approx \eta\alpha^2 m_p c^2$, with $\eta \simeq 0.1$. This quantity $\epsilon_{\rm nucl}$ sets the scale for triggering nuclear reactions in astrophysical contexts.

1.2.5 Gravitational Binding Energy

In the nonrelativistic, Newtonian theory for gravity, the gravitational energy of a system of size R and mass M will be $E_{\rm grav} \approx GM^2/R \approx (Gm_p^2/R)N^2$. This is not extensive with respect to N (for a given R), and the potential energy per

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particle varies as

$$\epsilon_g \equiv \frac{E_{\rm grav}}{N} = \left(\frac{Gm_p^2}{R}\right)N = \left(\frac{4\pi}{3}\right)^{1/3} Gm_p^2 N^{2/3} n^{1/3}, \tag{1.7}$$

where $n = (3N/4\pi R^3)$ is the number density of particles. The pressure due to gravitational force near the center of the object will be approximately

$$P_g \approx \frac{(GM^2/R^2)}{(4\pi R^2)} \approx \frac{1}{3}\left(\frac{4\pi}{3}\right)^{1/3} Gm_p^2 N^{2/3} n^{4/3} \cong \frac{1}{3}\left(\frac{E_{\rm grav}}{V}\right).$$

If the gravitational potential energy is comparable with the rest-mass energy of the system, it is necessary to take general relativistic effects into account. The ratio $\mathcal{R}_{gm} \equiv (E_{\rm grav}/E_{\rm mass})$ is $\mathcal{R}_{gm} \simeq 0.7(M/10^{33}\ {\rm gm})\,(R/1\ {\rm km})^{-1}$, which shows that if massive objects (with $M \simeq 10^{33}$ gm) are confined to small regions (with $R \simeq 1$ km), the system will exhibit general relativistic effects. When this ratio is small compared with unity, the system can be treated by Newtonian gravity.

1.2.6 Thermal and Degeneracy Energies of Particles

So far we have not introduced the notion of Temperature or the kinetic energy of the particle. These attributes bring in the next set of energy scales into the problem. For a particle of momentum p and mass m, the kinetic energy is given by

$$\epsilon = \sqrt{p^2c^2 + m^2c^4} - mc^2 = \begin{cases} p^2/2m & (p \ll mc) \\ pc & (p \gg mc) \end{cases}, \tag{1.8}$$

where the two forms are applicable in the non-relativistic (NR) and extreme relativistic (ER) limits. The behaviour of the system depends on the origin of the momentum distribution of the particles.

The familiar situation is the one in which short-range interactions (usually called 'collisions') between the particles effectively exchange the energy so as to randomize the momentum distribution. This will happen if the effective mean free path of the system l is small compared with the length scale L at which physical parameters change. (The explicit form taken by the condition $l \ll L$ can be very different in different cases; this condition is discussed in detail towards the end of this section.) When such a system is in steady state, we can assume that the local thermodynamic equilibrium, characterized by a local temperature T, exists in the system. Then the probability for occupying a state with energy E will scale as $P(E) \propto \exp[-(E/k_BT)]$. The typical momentum of the particle when the temperature is T is given by Eq. (1.8) with $\epsilon \simeq k_BT$, that is,

$$p \cong mc\left[\frac{2k_BT}{mc^2} + \left(\frac{k_BT}{mc^2}\right)^2\right]^{1/2} \cong \begin{cases} (2mk_BT)^{1/2} & (k_BT \ll mc^2;\ \text{NR}) \\ (k_BT/c) & (k_BT \gg mc^2;\ \text{ER}) \end{cases}. \tag{1.9}$$

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In this case, the momentum and the kinetic energy of the particles vanish when $T \to 0$.

The situation is actually more complicated for material particles like electrons. The mean energy of a system of electrons will not vanish even at zero temperature because electrons obey the Pauli exclusion principle, which requires that the maximum number of electrons that can occupy any quantum state be two, one with spin up and another with spin down. Because the uncertainty principle requires that $\Delta x \Delta p_x \gtrsim h$, we can associate $(d^3x d^3 p)/(2\pi\hbar)^3$ microstates with a phase-space volume $d^3x\, d^3 p$. Therefore the number of quantum states with momentum less than p is $[V(4\pi p^3/3)/(2\pi\hbar)^3]$, where V is the spatial volume available for the system. The lowest energy state will be the one in which the N electrons fill all levels up to some momentum p_F, called Fermi momentum. This requires that

$$n = \left(\frac{N}{V}\right) = 2\frac{(4\pi p_F^3/3)}{(2\pi\hbar)^3} = \frac{1}{3\pi^2}\left(\frac{p_F}{\hbar}\right)^3, \tag{1.10}$$

giving $p_F = \hbar(3\pi^2 n)^{1/3}$. It is obvious that if $p_F \gtrsim mc$ the system must be treated as relativistic, even in the zero-temperature limit. The energy corresponding to p_F will be

$$\epsilon_F = \sqrt{p_F^2c^2 + m^2c^4} - mc^2 = \begin{cases} \dfrac{p_F^2}{2m} = \left(\dfrac{\hbar^2}{2m}\right)(3\pi^2 n)^{2/3} & \text{(NR)} \\ p_F c = (\hbar c)(3\pi^2 n)^{1/3} & \text{(ER)} \end{cases}. \tag{1.11}$$

The quantity ϵ_F (called the Fermi energy) sets the quantum-mechanical scale of the energy; quantum-mechanical effects will be dominant if $\epsilon_F \gtrsim k_B T$ (degenerate), and the classical theory will be valid for $\epsilon_F \ll k_B T$ (nondegenerate). The relevant ratio $\mathcal{R}_{\rm ft} \equiv (\epsilon_F/k_B T)$ that determines that the degree of degeneracy is

$$\begin{aligned} \left(\frac{\epsilon_F}{k_B T}\right) &= \frac{mc^2}{k_B T}\left\{\left[\left(\frac{\hbar n^{1/3}}{mc}\right)^2 (3\pi^2)^{2/3} + 1\right]^{1/2} - 1\right\} \\ &\cong \begin{cases} \dfrac{1}{2}(3\pi^2)^{2/3}\left(\dfrac{\hbar^2}{m}\dfrac{n^{2/3}}{k_B T}\right) \\ (3\pi^2)^{1/3}\left(\dfrac{\hbar c}{k_B T} n^{1/3}\right) \end{cases}, \end{aligned} \tag{1.12}$$

where the two limiting forms are valid for $n \ll (\hbar/mc)^{-3}$ (NR) and $n \gg (\hbar/mc)^{-3}$ (ER), respectively. In the first case $[n \ll (\hbar/m_e c)^{-3} \simeq 10^{31}\ \text{cm}^{-3}]$, the system is nonrelativistic; it will also be degenerate if $\mathcal{R}_{\rm ft} = (\epsilon_F/k_B T) \gg 1$ and classical if $\mathcal{R}_{\rm ft} \ll 1$. The transition occurs at $\mathcal{R}_{\rm ft} \approx 1$, which corresponds to $nT^{-3/2} = [(mk_B)^{3/2}/\hbar^3] = 3.6 \times 10^{16}$ in cgs units. In the second case $[n \gg (\hbar/m_e c)^{-3} \simeq 10^{31}\ \text{cm}^{-3};\ \rho \equiv m_p n \gg 10^7\ \text{gm cm}^{-3}]$, electrons have $p_F \gg m_e c$ and are

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relativistic irrespective of temperature. The quantum effects will dominate thermal effects if $k_B T \ll (\hbar c) n^{1/3}$, and we will have a relativistic, degenerate gas.

In general, the kinetic energy of the particle will have contributions from the temperature as well as from Fermi energy. If we are interested in only the asymptotic limits, we can take the total kinetic energy per particle to be $\epsilon \approx \epsilon_F(n) + k_B T$. Note that such a system has a minimum energy $N\epsilon_F(n)$ even at $T = 0$.

By using our general result $P \simeq n\epsilon$ [see Eq. (1.3)], we can obtain the equation of state for the different cases discussed above. First, for a quantum-mechanical gas of fermionic particles with $k_B T \ll \epsilon_F$ and $\epsilon \approx \epsilon_F$, it follows from Eq. (1.11) that $P \simeq n\epsilon_F$ varies as the (5/3)rd power of density in the nonrelativistic case and as the (4/3)rd power of density in the relativistic case. Whether the system is relativistic or not is decided by the ratio (p_F/mc) or – equivalently – the ratio (ϵ_F/mc^2). The transition occurs at $n = n_{\text{RQ}} \approx (\hbar/mc)^{-3}$. Second, if the system is classical with $k_B T \gg \epsilon_F$ so that $\epsilon \cong k_B T$, then $P \simeq n k_B T$ in both nonrelativistic and extreme relativistic limits.

The energy scale of the individual particles also characterizes the energy involved in the collisions between the particles. If this quantity is larger than the binding energy of the atomic system, the atoms will be ionised and the electrons will be separated from the atoms. The familiar situation in which this happens is at high temperatures with $k_B T \gtrsim \epsilon_a$ when the system will be made of free electrons and positively charged ions, whereas, if $k_B T \ll \epsilon_a$, the system will be neutral. The transition temperature at which nearly half the number of atoms are ionised occurs around $k_B T \approx (\epsilon_a/10)$, which is $\sim 10^4$ K for hydrogen. For $T \gg 10^4$ K, the kinetic energy of the free electrons in the hydrogen plasma will be $\sim k_B T$.

The electrons can be stripped off the atoms in another different context. This occurs if the matter density is so high that the atoms are packed close to each other, with the electrons forming a common pool with $\epsilon_F \gtrsim \epsilon_a$. In this case, the electrons will be quantum mechanical and the relevant energy scale for them will be ϵ_F. The temperature does not enter into the picture if $k_B T \ll \epsilon_F$, and we may call this a zero-temperature plasma. Conventionally, such systems are called degenerate. For normal metals in the laboratory the Fermi energy is comparable with the binding energy within an order of magnitude. If the temperature is below 10^4 K, the properties of the system are essentially governed by Fermi energy.

In the derivation of P in Eq. (1.3) it is assumed that the the gas is ideal, i.e., the mutual interaction energy of the particles is small compared with the kinetic energy. To treat a plasma as ideal, it is necessary that the Coulomb interaction energy of ions and electrons be negligible. The typical Coulomb potential energy between the ions and the electrons in the plasma is given by $\epsilon_{\text{Coul}} \approx Zq^2 n^{1/3}$. If the classical high-temperature plasma is to be treated as an ideal gas, this energy should be small compared with the energy scale of the particle $\epsilon \approx k_B T$, which requires the condition $nT^{-3} \ll (k_B/Zq^2)^3 \simeq 2.2 \times 10^8 Z^{-3}$ in cgs units. On the other hand, to treat the high-density quantum gas as ideal, we should require that

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the Coulomb energy $\epsilon_{\rm Coul} \approx Zq^2 n^{1/3}$ be small compared with the Fermi energy $\epsilon_F \approx (\hbar^2/2m)n^{2/3}$. The condition now becomes $n \gg 8Z^3 a_0^{-3} \approx Z^3 \times 10^{26}\ \rm cm^{-3}$. Note that such a system becomes more ideal at higher densities; this is because the Fermi energy rises faster than the Coulomb energy.

Let us now go back to the tacit assumption we made in the above analysis, viz., that physical interactions between the particles of the system are capable of maintaining the thermal equilibrium. Determining the precise condition that will ensure this is not a simple task; but – naively – we would require that (1) the mean free path for particles, $l = (n\sigma)^{-1}$ based on a relevant scattering process governed by a cross section σ, be small compared with the scale L over which various parameters change significantly, and (2) that the mean time between collisions $\tau = (nv\sigma)^{-1}$ be small compared with the time scale over which physical parameters change.

To apply this condition we need to know the relevant mean free path for the system. For a neutral gas of molecules, this is essentially determined by molecular collisions with $\sigma_0 \approx \pi a_0^2 \approx 8.5 \times 10^{-17}\ \rm cm^2$ and $l = (n\sigma_0)^{-1}$. The time scale for the establishment of a Maxwellian distribution of velocities will be approximately $\tau_{\rm neu} \simeq l/v \propto n^{-1}T^{-1/2}$. For an ionized classical gas, the cross section for scattering is decided by Coulomb interaction between charged particles. Because an ionized plasma is made of electrons and ions with vastly different inertia, the interparticle collisions can take different time scales to produce thermal equilibrium between electrons, between ions, and between electrons and ions. Each of these needs to be discussed separately.

The typical impact parameter between two electrons is $b \approx (2Zq^2/m_e v^2)$, where v is the typical velocity of an electron. The corresponding e–e scattering cross section is

$$\sigma_{\rm coul} \approx \pi b^2 \approx \pi \left(\frac{Zq^2}{m_e}\right)^2 \frac{1}{v^4} \approx 10^{-20}\ {\rm cm^2}\ Z^2 \left(\frac{T}{10^5\ \rm K}\right)^{-2}, \tag{1.13}$$

and the mean free path varies as $l = (n\sigma_{\rm coul})^{-1} \propto (T^2/n)$. The mean free time between the electron–electron scattering will be $\tau_{ee} \approx (n\sigma v)^{-1}$, where n is the number density of electrons and $\sigma \approx \pi b^2$. This gives $\tau_{ee} \approx (m_e^2 v^3/2\pi Z^2 q^4 n)$, which is the leading dependence. (A more precise analysis changes the numerical coefficient and introduces an extra logarithmic factor; see Chap. 9.)

Note that $\tau \propto m^2 v^3 \propto T^{3/2} m^{1/2}$ at a given temperature $T \propto (1/2)mv^2$. Therefore the ion–ion collision time scale τ_{pp} will be larger by the factor $(m_p/m_e)^{1/2} \simeq 43$, giving $\tau_{pp} = (m_p/m_e)^{1/2}\tau_{ee} \simeq 43\tau_{ee}$.

The time scale for significant transfer of energy between electrons and ions is still larger because of the following fact. When two particles (of unequal mass) scatter off each other, there is no energy exchange in the centre-of-mass frame. In the case of ions and electrons, the centre-of-mass frame differs from the lab frame only by a velocity $v_{\rm CM} \simeq (m_e/m_p)^{1/2} v_p \ll v_p$. Because there is

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no energy exchange in the centre-of-mass frame, the maximum energy transfer in the lab frame (which occurs for a head-on collision) is approximately $\Delta E = (1/2)m_p(2v_{\rm cm})^2 = 2m_p v_{\rm cm}^2 \simeq 2m_e v_p^2$, giving $[\Delta E/(1/2)m_p v_p^2] \simeq (m_e/m_p) \ll 1$. Therefore it takes (m_p/m_e) times more collisions to produce equilibrium between electrons and ions, that is, the time scale for electron–ion collision is $\tau_{pe} = (m_p/m_e)\tau_{ee} \simeq 1836\tau_{ee}$. The plasma will relax to a Maxwellian distribution in this time scale.

Finally, it must be noted that in a high-temperature tenuous plasma, this mean free path can become larger than the size of the system. If that happens, it is necessary to check whether there are any other physical processes that can provide an effective mean free path that is lower. Most astrophysical plasmas host magnetic fields that make the charged particles spiral around the magnetic-field lines. We can estimate the typical radius of a spiraling charged particle in a magnetic field by equating the centrifugal force (mv^2/r) to the magnetic force (qvB/c). This leads to a radius called the Larmor radius, given by

$$r_L = (mcv/qB) = 13 \text{ cm } (T/10^5 \text{ K})^{1/2}(B/1 \text{ G})^{-1}$$

in a thermal plasma. When the Larmor radius is small, it can act as the effective mean free path for the scattering of charged particles. The ratio between the mean free path from Coulomb collisions $[l \propto (T^2/n)]$ and the Larmor radius $[r_L \propto (T^{1/2}/B)]$ varies as $(BT^{3/2}/n)$ and can be large in tenuous high-temperature plasmas with strong magnetic fields. This ratio is unity for a critical magnetic field:

$$B_c = 10^{-19} \text{ G}\left(\frac{T}{10^5 \text{ K}}\right)^{-3/2}\left(\frac{n}{1 \text{ cm}^{-3}}\right). \tag{1.14}$$

The magnetic field in most astrophysical plasmas will be larger than B_c, and hence this effect will be important.

1.3 Classical Radiative Processes

We next turn to the question of gathering information about the cosmic structures from the radiation received from them. To relate the information received through the electromagnetic waves to the properties of the emitting system, it is necessary to understand the process of electromagnetic radiation from different systems and the nature of the spectrum emitted by each of them.

In classical electromagnetic theory, radiation is emitted by any charged particle that is in accelerated motion. A detailed argument given in Chap. 3 shows that the total amount of energy radiated per second in all directions by a particle with charge q moving with acceleration a is given by

$$\frac{d\mathcal{E}}{dt} = \frac{2}{3}\frac{q^2}{c^3}a^2, \tag{1.15}$$

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