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Introduction

A Gaussian Hilbert space is a (complete) linear space of random variables with (centred) Gaussian distributions. This simple notion combines probability theory and Hilbert space theory into a rich and powerful structure, and Gaussian Hilbert spaces and connected notions such as the Wiener chaos decomposition and Wick products appear in several areas of probability theory and its applications, for example in stochastic processes and fields, stochastic integration, quantum field theory and limit theory for various statistics. There are also applications to non-probabilistic analysis, for example Banach space geometry and partial differential equations.

Although there are many references dealing with such applications where Gaussian spaces are treated and used, see for example Hida and Hitsuda (1976), Hida, Kuo, Potthoff and Streit (1993), Holden, Øksendal, Ubøe and Zhang (1996), Ibragimov and Rozanov (1970), Kahane (1985), Kuo (1996), Major (1981), Malliavin (1993, 1997), Meyer (1993), Neveu (1968), Nualart (1995, 1997+), Obata (1994), Pisier (1989), Simon (1974, 1979a), Watanabe (1984), there seems to be a shortage of works dealing with the basic properties of Gaussian spaces in general, without connecting them to a particular application. (One exception is the paper by Dobrushin and Minlos (1977).) This book is an attempt to fill the gap by providing a collection of the most important definitions and results for general Gaussian spaces, together with some applications to special Gaussian spaces. For further results and applications, see the references above.

In our point of view, the situation is similar to that of Hilbert spaces in functional analysis. All spaces (Hilbert or Gaussian, respectively) of the same dimension are isomorphic, so from an abstract point of view it suffices to study one of them. Nevertheless, there are many different concrete examples of such spaces that arise in different contexts, and the importance (and perhaps beauty) of the general theorems is revealed by their interpretations for the various examples. Hence, on the one hand, it seems best to develop the theory of Gaussian spaces abstractly, formulating definitions and results in terms of the intrinsic structure only. On the other hand, different Gaussian spaces may then be used to illustrate and apply the general results.

We take a probabilistic point of view, regarding the basic objects as random variables. In accordance with tradition in probability theory, we will therefore assume these variables to be defined on some probability space, but we will not make any further assumptions on this probability space, and we will usually not consider it explicitly. (In some examples, however, we use specific probability spaces for special purposes.)

The core of the book consists of Chapters 1–5, where the central parts of the theory are developed. Gaussian Hilbert spaces, and some related notions, are defined in Chapter 1, where also many examples (both simple and less simple) are given. Moreover, we introduce Feynman diagrams as a convenient bookkeeping method in moment calculations.

Every Gaussian Hilbert space induces an orthogonal decomposition, known as the (Wiener) chaos decomposition, of the corresponding L^2-space of all square integrable random variables that are measurable with respect to the σ-field generated by the Gaussian Hilbert space. This decomposition is introduced and studied in Chapter 2.

The chaos decomposition further forms the basis for the definition of Wick products in Chapter 3. This chapter also includes the definition of Wick exponentials, which form a simple family of random variables that is useful on many occasions.

It is shown in Chapter 4 that the Wiener chaos decomposition and the Wick products can be regarded as a concrete realization of the symmetric tensor products of the Gaussian Hilbert space. This point of view leads to new results; in particular, a contractive linear map of one Gaussian Hilbert space into another extends in a canonical way to a linear contraction between the corresponding L^2-spaces. This mapping is further shown to be a contraction on L^p for every $p \geq 1$. An important example of a mapping that is obtained as such an extension is the Mehler transform.

Chapter 5 is a continuation of Chapter 4, treating hypercontractivity, i.e. the property that the operators defined in Chapter 4 under suitable conditions are contractions from one L^p-space into another with a different exponent. This chapter is perhaps more technical than the preceding ones; on the other hand, it contains powerful theorems that have rather deep applications.

The remaining chapters present various applications or further developments. They build upon the first chapters, but are to a large extent independent of each other, and the reader may choose rather freely among them according to his or her interests. The main purpose of these chapters is to show how the general theory is used in different contexts. We do not intend to give complete coverage of the topics studied there; on the contrary, we concentrate on results that are directly related to the main theme of this book, and we try to avoid going too deep. The reader who has a special interest in some of these applications will certainly need other sources for further results, but we hope that the present book may serve as a useful introduction or complement.

Chapter 6 studies the distribution of random variables with a finite chaos decomposition; in particular, variables with no terms beyond the second order are studied in detail. (Such variables occur frequently in limit theorems of the type studied in Chapter 11.)

Chapter 7 contains some important applications to stochastic integration, beginning with the usual Itô integral with respect to Brownian motion and then introducing some extensions; in particular the Gaussian stochastic integral over general measure spaces and the Skorohod integral.

In Chapter 8 we study a few aspects of Gaussian stochastic processes, emphasizing the Hilbert space geometrical point of view. We further define and study the Cameron–Martin space, which is a Hilbert space of (deterministic) functions on the index set associated to a Gaussian stochastic process.

In Chapter 9 we give some simple results on conditioning in Gaussian spaces. The results are applied in an example treating a random Gaussian potential in a network.

Chapter 10 studies pairs of subspaces of a Gaussian Hilbert space. The relation between two subspaces is described by a projection operator and a corresponding sequence of numbers, and it is shown how various measures of dependence between the two subspaces can be estimated using these numbers.

In Chapter 11 we give applications to results on the asymptotic distributions of U-statistics and related random variables, some of which appear in the study of random graphs. It is noteworthy that in this chapter, unlike the preceding ones, the Gaussian Hilbert spaces are not present in the problem from the beginning; they are introduced as a convenient technical tool for the solution.

Chapter 12 contains applications to operator theory related to Grothendieck's theorem. In particular, both an upper and a lower bound to Grothendieck's constant are given. Also in this chapter, the Gaussian Hilbert spaces are introduced as a technical tool.

In the remaining chapters we return to the study of general Gaussian Hilbert spaces. In Chapter 13 we define and study the annihilation, creation, position and momentum operators that are important in quantum mechanics. (No physics is assumed or explained; our treatment is purely mathematical.) In connection with this we also present Wick's original definition of the Wick product, using 'Wick ordering', and show how it relates to the Wick product for Gaussian Hilbert spaces.

In Chapter 14 we study an operation on random variables generalizing the shift of a Brownian motion studied by Cameron and Martin (1944).

Chapter 15 contains an introduction to Malliavin calculus, based on results in Chapters 13 and 14. In particular, we give a detailed treatment of Gaussian Sobolev spaces and the Meyer inequalities, and results on existence and smoothness of densities. We also give another interpretation of the Skorohod integral.

Finally, Chapter 16 introduces some transforms that map random variables to continuous functions on a suitable space (typically the Gaussian Hilbert space itself). These transforms are useful in several contexts, but we have chosen to put them in the last chapter and merely give some examples of their use rather than introducing and using them earlier. One application of them is a more general definition of the Wick product than the one given in Chapter 3. Another application is a new definition and extension of the Skorohod integral.

Some background material from probability theory and functional analysis is presented in Appendices A–H. (Thus, for example, 'Theorem A.1' and 'equation (C.1)' refer to the appendices.)

A graduate course could be based on the first five chapters together with a selection of material from later chapters. Suitable prerequisites for this book are a standard course or two in probability theory, some integration theory and some functional analysis, especially elementary Hilbert space theory.

The selection of material and methods presented here is, of course, partly a matter of taste. For example, when deriving the basic properties in the first chapters, we often prefer to use Feynman diagrams and simple combinatorics rather than generating functions, and we avoid using properties of Hermite polynomials that have to be verified by other methods (instead we show that many of these properties follow from the probabilistic results).

The results in this book are collected from many different sources, and although some proofs are our own, the original references are seldom given and absence of references does not imply that the results are new.

I would like to thank the many other mathematicians who have contributed with helpful discussions, references and corrections; in particular I would like to mention Pontus Andersson, Persi Diaconis, Allan Gut, Sten Kaijser, Paul Malliavin, Bernt Øksendal, Gunnar Peters, Jim Propp and Philip Protter.

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My daughter Sofie was born while I was completing this book. Any remaining inconsistencies or errors are entirely due to her distracting influence.

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