I

Gaussian spaces

1. Preliminaries

We recall some standard notions from probability theory, integration theory and functional analysis, at the same time fixing some basic notation.

A real or complex *random variable* is a measurable function on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. We usually allow complex variables without further comment; in many cases there will be no difference between the real and complex cases, and we then allow both possibilities. (Complex variables are important in some applications but not used in others.)

We will occasionally also consider random variables with values in some other space E, for example vector-valued variables, but we will then always say so explicitly. (We will only be interested in cases when E is a topological space equipped with its Borel σ-field, which we denote by $\mathcal{B}$.)

We identify random variables that are equal a.s.; hence we usually write $X = Y$ rather than the longer $X = Y$ a.s., but these expressions are equivalent. In fact, many constructions will give us variables that are uniquely defined only a.s. (Formally, random variables may be defined as equivalence classes of measurable functions, but we will not stress this point of view.)

The underlying probability space $(\Omega, \mathcal{F}, \mathrm{P})$ is completely arbitrary and is often not even mentioned. Of course, it is understood that whenever we talk about joint distributions or sums of random variables, they have to be defined on the same probability space. In particular, a linear space of random variables (the main theme of this book) is a linear space of measurable functions on some probability space. We will tacitly use $(\Omega, \mathcal{F}, \mathrm{P})$ to denote the underlying probability space of any random variable or space of random variables under discussion, as long as there is no danger of confusion.

Given a set A of random variables on some probability space $(\Omega, \mathcal{F}, \mathrm{P})$, we let $\mathcal{F}(A)$ denote the σ-field generated by the variables in A; this is the smallest σ-field such that all variables in A are measurable. Clearly, $\mathcal{F}(A) \subseteq \mathcal{F}$.

The *distribution* of a random variable X with values in some space E is the probability measure on E induced by $X\colon (\Omega, \mathcal{F}, \mathrm{P}) \to E$. We write $X \overset{\mathrm{d}}{=} Y$ if X and Y have the same distribution, and denote convergence in distribution (i.e. convergence of the corresponding distributions in the usual sense) by $\overset{\mathrm{d}}{\to}$.

The L^p-norm of a random variable X is defined by

$$\|X\|_p = (\mathrm{E}\,|X|^p)^{1/p}$$

for $0 < p < \infty$, for $p = \infty$ this is modified to $\|X\|_\infty = \operatorname{ess\,sup}|X|$. (This is really a norm only when $1 \le p \le \infty$, since the triangle inequality fails for $0 < p < 1$.) For $0 < p \le \infty$, $L^p = L^p(\Omega, \mathcal{F}, \mathrm{P})$ is the linear space of all random variables X defined on $(\Omega, \mathcal{F}, \mathrm{P})$ such that $\|X\|_p < \infty$.

We will mainly use L^2, which is a Hilbert space with the inner product $\langle X, Y\rangle = \mathrm{E}(X\overline{Y})$; thus $\|X\|_2^2 = \langle X, X\rangle$. Note that if X and Y are real random variables with expectations 0, then $\|X\|_2^2 = \mathrm{Var}(X)$ and $\langle X, Y\rangle = \mathrm{Cov}(X, Y)$. In particular, two such variables are orthogonal if and only if they are uncorrelated. (Convergence in L^2 of random variables is traditionally known as *convergence in mean square.*)

L^p is a Banach space for $1 \le p \le \infty$. If $1 \le p < \infty$, the dual space of L^p equals $L^{p'}$, where p' is the conjugate exponent defined by $1/p + 1/p' = 1$.

We further use $L^0 = L^0(\Omega, \mathcal{F}, \mathrm{P})$ to denote the space of all random variables on $(\Omega, \mathcal{F}, \mathrm{P})$, equipped with the topology of convergence in probability. This is a complete metric topological vector space, where the metric may be chosen as $\mathrm{E}(|X - Y| \wedge 1)$; for further topological properties of this space see for example Dunford and Schwartz (1958, Section IV.11, where the space is denoted by TM).

Convergence in probability is denoted $\xrightarrow{\mathrm{p}}$.

Recall that (for a probability space) the L^p-spaces decrease as p increases; if $0 \le p \le q \le \infty$, then $L^p \supseteq L^q$. More precisely,

$$\|X\|_p \le \|X\|_q, \qquad 0 < p \le q \le \infty,$$

which is known as *Lyapounov's inequality*. Furthermore, L^q is a dense subspace of L^p when $0 \le p \le q \le \infty$, because truncations of any random variable $X \in L^p$ converge to X in L^p.

As stated above, we allow complex random variables. We usually make no distinction in the notation between the real and complex cases (assuming both are valid); for example, L^p stands for both the real spaces of real variables and the corresponding complex spaces of complex variables. When it is necessary to distinguish between spaces of real and complex random variables, we write $L^p_{\mathbb{R}}$ and $L^p_{\mathbb{C}}$, respectively. We use these subscripts in the same way also for various spaces of functions on $\mathbb{R}$ or on other sets.

A set in a topological vector space is *total* if the family of (finite) linear combinations of elements of the set is dense in the space. For a Banach space (or more generally a locally convex space), this is equivalent to requiring that every continuous linear functional that vanishes on the set vanishes identically.

We let $\mathbf{1}[S]$ denote the indicator function of a statement S, which is defined to be 1 if the statement holds and 0 otherwise. Similarly, if A is a set, $\mathbf{1}_A(x) = \mathbf{1}[x \in A]$ is the function that equals 1 when $x \in A$ and 0 when $x \notin A$. (This is known as the indicator function of A, and outside probability theory also as the characteristic function of A.)

2. Gaussian random variables

In this book a *Gaussian* or *normal* random variable is a real-valued random variable with characteristic function $\exp(i\mu t - \frac{1}{2}\sigma^2 t^2)$ for some $\mu \in (-\infty, \infty)$ and $\sigma^2 \geq 0$. Note that we include the degenerate case $\sigma^2 = 0$, when the variable a.s. equals μ. The variable is *centred* or *symmetric* if $\mu = 0$, and *standard* if $\mu = 0$ and $\sigma^2 = 1$. We denote the distribution of a Gaussian variable by $\mathrm{N}(\mu, \sigma^2)$, with μ and σ^2 as above.

We assume that the reader is familiar with the basic properties of normal distributions, for example that μ is the mean and σ^2 the variance of the variable, that all moments are finite, and that if $\sigma^2 > 0$, then the normal distribution has the density $(2\pi\sigma^2)^{-1/2} \exp\bigl(-(x-\mu)^2/2\sigma^2\bigr)$, $-\infty < x < \infty$; moreover, if $X \sim \mathrm{N}(\mu, \sigma^2)$, then $\mathrm{E}\, e^{zX} = \exp\left(\mu z + \frac{1}{2}\sigma^2 z^2\right)$ for any complex number z.

A finite number of random variables $\xi_1, \dots, \xi_n$ are said to have a *joint normal distribution* if $\sum_1^n t_i \xi_i$ has a normal distribution for any real numbers $t_1, \dots, t_n$. We say that an infinite set of random variables has a joint normal distribution if every finite subset has.

The joint characteristic function of a finite set $\xi_1, \dots, \xi_n$ of jointly normal variables is $\mathrm{E} \exp(i \sum_j t_j \xi_j) = \exp\bigl(i \sum_j t_j \, \mathrm{E}\, \xi_j - \frac{1}{2} \sum_{j,k} t_j t_k \operatorname{Cov}(\xi_j, \xi_k)\bigr)$. Hence the distribution of a finite set of jointly normal variables is determined by their means and covariances; this extends to infinite sets by a standard argument, see Example A.3.

In particular, two or more jointly normal variables are independent if (and only if) their covariances vanish. For centred variables, this is equivalent to the variables being orthogonal.

The Gaussian variables considered in this book will almost always be centred. This is not very restrictive, since any Gaussian variable can be written as the sum of a centred Gaussian variable and a constant.

We will occasionally also use complex or vector-valued Gaussian variables. We say that the random vector $\xi = (\xi_1, \dots, \xi_n)$ has a normal (or Gaussian) distribution in $\mathbb{R}^n$ if $\xi_1, \dots, \xi_n$ have a joint normal distribution, i.e. if $\xi \cdot t$ is normal for every vector $t \in \mathbb{R}^n$. (This notion can be extended to random variables in infinite-dimensional vector spaces, see Example 1.13 below.) Equivalently, a random vector ξ in $\mathbb{R}^n$ is normal if and only if its characteristic function equals $\exp(i\mu \cdot t - \frac{1}{2} t' \Sigma t)$ for some vector $\mu \in \mathbb{R}^n$ (the mean) and some semi-definite matrix Σ (the covariance matrix).

We similarly say that a complex random variable ζ is Gaussian if its real and imaginary parts $\operatorname{Re} \zeta$ and $\operatorname{Im} \zeta$ have a joint normal distribution. We return to further comments on complex Gaussian variables in Section 4.

Note that although general random variables are allowed to be complex in this book, Gaussian variables are always real unless we explicitly consider complex (or vector-valued) Gaussian variables.

REMARK 1.1. Although these definitions are standard, the reader should note that some authors prefer slight variations; for example, some define Gaussian random variables to be symmetric. We find it less confusing (although somewhat tedious) instead to repeat 'symmetric' or 'centred' whenever necessary. Also, we use the expressions 'Gaussian variable' and 'normal variable' as synonyms, without attaching any significance to the choice of one or the other.

3. Gaussian Hilbert spaces

DEFINITION 1.2. A *Gaussian linear space* is a real linear space of random variables, defined on some probability space $(\Omega, \mathcal{F}, \mathrm{P})$, such that each variable in the space is centred Gaussian. Obviously, a Gaussian linear space is a linear subspace of $L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mathrm{P})$, and we use the norm and inner product of L^2 on it. A *Gaussian Hilbert space* is a Gaussian linear space which is complete, i.e., a closed subspace of $L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mathrm{P})$ consisting of centred Gaussian random variables.

Note that we require all random variables in a Gaussian space to be *real*. We will give some comments on linear spaces of complex Gaussian variables in the next section, but we will almost exclusively study the real case. On the other hand, we will often study complex random variables (Gaussian or non-Gaussian) constructed from the real variables in a Gaussian space; this can usually be done without any complications, and it is important for many applications.

Note also that we require the variables in a Gaussian space to be *centred* Gaussian; this is important for the properties developed in the sequel and spaces consisting of general Gaussian variables seem to be of much less use. If one for some reason is given a linear space V of general (real) Gaussian variables, one can always study the Gaussian space $G = \{\xi - \mathrm{E}\,\xi : \xi \in V\}$, and use the inclusion $V \subseteq G \oplus \mathbb{R}$ to transfer results to V.

A Gaussian linear space can always be completed to a Gaussian Hilbert space.

THEOREM 1.3. *If $G \subset L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mathrm{P})$ is a Gaussian linear space, then its closure $\overline{G}$ in L^2 is a Gaussian Hilbert space.*

PROOF. Suppose that $\xi \in \overline{G}$. We have to show that ξ has a centred normal distribution.

There exists a sequence $\xi_n \in G$ such that $\xi_n \to \xi$ in L^2. Let $\sigma^2 = \|\xi\|_2^2 = \mathrm{E}\,\xi^2$ and $\sigma_n^2 = \|\xi_n\|_2^2$. Then $\sigma_n^2 \to \sigma^2$ as $n \to \infty$. Convergence in L^2 implies convergence in distribution, and since $\xi_n \sim \mathrm{N}(0, \sigma_n^2) \xrightarrow{\mathrm{d}} \mathrm{N}(0, \sigma^2)$ as $n \to \infty$, we obtain $\xi \sim \mathrm{N}(0, \sigma^2)$. □

In view of this result, there is no real loss of generality in considering only Gaussian Hilbert spaces, and we will do so in much of the sequel. Nevertheless, incomplete Gaussian spaces appear naturally, as in several of the examples below, and it may sometimes be of interest to consider them as they are instead of immediately completing them.

Since Gaussian variables have moments of all orders, a Gaussian linear space is a subspace of L^p for every finite p.

THEOREM 1.4. *The L^p-norms, $0 < p < \infty$, are all proportional on a Gaussian linear space G. Hence all L^p-topologies coincide on G; they also coincide with the topology of convergence in probability.*

Moreover, the closure of G in any L^p, $0 \le p < \infty$, equals the Gaussian Hilbert space $\overline{G}$. In particular, a Gaussian Hilbert space is a closed subspace of every $L^p_{\mathbb{R}}$, $0 \le p < \infty$.

PROOF. A standard computation shows that if ξ is a centred Gaussian variable, then

$$\|\xi\|_p = (\mathrm{E}\,|\xi|^p)^{1/p} = \kappa(p)\|\xi\|_2, \qquad 0 < p < \infty, \tag{1.1}$$

where $\kappa(p) = \sqrt{2}\big(\Gamma(\frac{p+1}{2})/\sqrt{\pi}\big)^{1/p}$. Hence, a Cauchy sequence in one L^p-norm in G is also a Cauchy sequence in any other L^p-norm, which shows that the closures in the two topologies coincide.

Moreover, suppose that $(\xi_n)_1^\infty$ is a sequence in G such that $\xi_n \overset{\mathrm{p}}{\to} \xi$ as $n \to \infty$ for some random variable ξ. Then $\xi_m - \xi_n \overset{\mathrm{p}}{\to} 0$ as $m, n \to \infty$, and since the variables $\xi_m - \xi_n \in G$ are Gaussian, this implies $\|\xi_m - \xi_n\|_2 \to 0$. Consequently, $(\xi_n)_1^\infty$ is a Cauchy sequence in L^2. Thus the sequence converges in L^2 to some limit, which has to be ξ, and the remaining assertions follow. □

We have required that each variable in a Gaussian space has a normal distribution. It follows that the variables furthermore are jointly normal.

THEOREM 1.5. *Any set of random variables in a Gaussian linear space has a joint normal distribution.*

PROOF. By definition, it suffices to consider a finite set. Thus, let $\xi_1, \dots, \xi_n$ belong to a Gaussian linear space G. If $t_1, \dots, t_n$ are arbitrary real numbers, then $\sum_1^n t_i\xi_i \in G$, and thus $\sum_1^n t_i\xi_i$ has a normal distribution. This implies that $\xi_1, \dots, \xi_n$ has a joint normal distribution. □

We give some important examples of Gaussian spaces.

EXAMPLE 1.6. Let ξ be any non-degenerate, normal variable with mean zero. Then $\{t\xi : t \in \mathbb{R}\}$ is a one-dimensional Gaussian Hilbert space.

EXAMPLE 1.7. Let $\xi_1, \dots, \xi_n$ have a joint normal distribution with mean zero. Then their linear span $\{\sum_1^n t_i\xi_i : t_i \in \mathbb{R}\}$ is a finite-dimensional Gaussian Hilbert space. Conversely, by Theorem 1.5, every finite-dimensional Gaussian space is of this type.

EXAMPLE 1.8. More generally, if $\{\xi_\alpha\}$ is any set of centred jointly normal variables, then the linear span of $\{\xi_\alpha\}$ is a Gaussian linear space, and, by

Theorem 1.3, the closed linear span of $\{\xi_\alpha\}$ in $L^2_{\mathbb{R}}$ is a Gaussian Hilbert space. These spaces are called *the Gaussian linear space spanned by* $\{\xi_\alpha\}$ and *the Gaussian Hilbert space spanned by* $\{\xi_\alpha\}$, respectively.

EXAMPLE 1.9. Let $\{\xi_\alpha\}$ by any set (finite or infinite, possibly uncountable) of independent standard normal random variables. By the preceding example, their closed linear span

$$\Big\{\sum_\alpha a_\alpha \xi_\alpha : \sum_\alpha a_\alpha^2 < \infty\Big\}$$

is a Gaussian Hilbert space.

We observe that every Gaussian Hilbert space is of this type. In fact, let $\{\xi_\alpha\}$ be any orthonormal basis in the space. The variables ξ_α are uncorrelated, and thus independent, standard normal variables, and their closed linear span equals the given space. From an abstract point of view, all Gaussian Hilbert spaces of the same dimension are thus the same. Nevertheless, different concrete examples of Gaussian Hilbert spaces are useful for different applications, and we continue with some further examples.

EXAMPLE 1.10. Let B_t, $0 \le t < \infty$, be a standard Brownian motion. By Example 1.8, the closed linear span of $\{B_t\}_{t\ge 0}$ is a Gaussian Hilbert space, which we denote by $H(B)$. As will be seen in detail in Chapter 7, this space has a simple representation in terms of stochastic integrals, viz.

$$H(B) = \Big\{\int_0^\infty f(t)\,dB_t\Big\},$$

where f ranges over the set of (deterministic) functions in $L^2_{\mathbb{R}}([0,\infty), dt)$.

EXAMPLE 1.11. Similarly, if $(X_t)_{t\in T}$ is any Gaussian stochastic process in discrete or continuous time (T is a suitable subset of $\mathbb{R}$), and if (for simplicity) $\operatorname{E} X_t = 0$ for all t, then the closed linear span of $\{X_t\}$ is a Gaussian Hilbert space. By taking the closed linear span of e.g. $\{X_t\}_{t\le t_0}$ or $\{X_t\}_{t\ge t_0}$ instead, we obtain closed Gaussian subspaces of this space.

EXAMPLE 1.12. More generally, we may, for any set T, define a Gaussian stochastic process indexed by T to be a family X_t, $t \in T$, of jointly normal random variables. (This simple definition is sufficient for our purposes; as is well-known, it is not well suited for studying pathwise properties such as sample path continuity when T is uncountable; see Appendix B.) Again, if the process is centred, i.e. $\operatorname{E} X_t = 0$ for all t, then the closed linear span of $\{X_t\}_{t\in T}$ is a Gaussian Hilbert space. Hence a centred Gaussian stochastic process indexed by T is the same as a function from T into some Gaussian Hilbert space.

We study this example in some detail in Chapter 8.

EXAMPLE 1.13. Let $\mathcal{X}$ be a real Banach space, or more generally, a locally convex topological vector space. A Borel probability measure μ on $\mathcal{X}$

is said to be Gaussian if each continuous linear functional $x^* \in \mathcal{X}^*$, regarded as a random variable defined on the probability space $(\mathcal{X}, \mathcal{B}, \mu)$, is Gaussian. If furthermore μ is symmetric, this means that $\mathcal{X}^*$ is a Gaussian space, which may be completed to a Gaussian Hilbert space contained in $L^2(\mathcal{X}, \mathcal{B}, \mu)$. (Some elements x^* may be 0 μ-a.e., in which case the Gaussian space is really a quotient space of $\mathcal{X}^*$.)

Similarly, a random variable ξ with values in $\mathcal{X}$ is said to be Gaussian if the real-valued random variable $\langle x^*, \xi \rangle$ is Gaussian for every $x^* \in \mathcal{X}^*$, or equivalently, if the distribution of ξ is a Gaussian measure on $\mathcal{X}$. If ξ is also symmetric, then the set $\{\langle x^*, \xi \rangle : x^* \in \mathcal{X}^*\}$ is a Gaussian linear space; if μ is the distribution of ξ, this space is naturally isomorphic to the Gaussian space just constructed by considering $\mathcal{X}^*$ as a space of random variables defined on $(\mathcal{X}, \mathcal{B}, \mu)$.

EXAMPLE 1.14. A simple special case of Example 1.13 is obtained by taking $\mathcal{X}$ to be a finite-dimensional Euclidean space $\mathbb{R}^d$. The Gaussian measures on $\mathbb{R}^d$ are just the d-dimensional Gaussian distributions, i.e. the distributions of d-dimensional Gaussian random vectors.

Given a centred Gaussian measure on $\mathbb{R}^d$, the Gaussian Hilbert space constructed in Example 1.13 is just the space of all linear functionals on $\mathbb{R}^d$. This space is d-dimensional, provided the measure is not singular.

An important example is the *standard d-dimensional Gaussian measure*, given by the density

$$(2\pi)^{-d/2} e^{-|x|^2/2};$$

this is just the product of d standard Gaussian measures on $\mathbb{R}$, and is therefore the distribution of $(\xi_1, \dots, \xi_d)$ with ξ_i independent standard normal.

EXAMPLE 1.15. Consider the probability space $(\mathbb{R}, \mathcal{B}, \gamma)$, where γ is the standard Gaussian measure $d\gamma = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$. Then $\xi = x$ is a standard normal random variable, and $H = \{tx : t \in \mathbb{R}\}$ is a one-dimensional Gaussian Hilbert space. (This is a special case of Example 1.6; it is also the simplest but perhaps most important case of Example 1.14.)

This example is an important bridge between the probabilistic theory of Gaussian Hilbert spaces and real analysis; we will several times obtain interesting results in real analysis by interpreting general theorems for this particular Gaussian Hilbert space.

EXAMPLE 1.16. (Rather long and technical, and may be skipped at the first reading.) An example of central importance in quantum field theory and elsewhere is obtained by specializing Example 1.13 to the space $S'(\mathbb{R}^d)$ of real tempered distributions, the dual of the space $S(\mathbb{R}^d)$ of real rapidly decreasing smooth functions (we omit the subscript $\mathbb{R}$ on S in this example).

Let φ denote the canonical embedding of $S(\mathbb{R}^d)$ into its bidual. (In fact, the space is reflexive so φ is an isomorphism.) Explicitly, if $f \in S(\mathbb{R}^d)$, then $\varphi(f)$ is the linear functional $u \mapsto u(f)$ defined on $S'(\mathbb{R}^d)$.

Suppose that μ is a symmetric Gaussian probability measure on $S'(\mathbb{R}^d)$. Then, by Example 1.13, $\varphi(f) \in L^2(S'(\mathbb{R}^d), \mu)$ is a symmetric Gaussian random variable for every $f \in S(\mathbb{R}^d)$. Let H be the closure in $L^2(\mu)$ of $\{\varphi(f) : f \in S(\mathbb{R}^d)\}$; thus H is a Gaussian Hilbert space.

Define the symmetric semi-definite bilinear form $\mathcal{E}$ on $S(\mathbb{R}^d)$ by

$$\mathcal{E}(f,g) = \langle \varphi(f), \varphi(g) \rangle_H = \operatorname{E} \varphi(f)\varphi(g).$$

If $f_n \to f$ in $S(\mathbb{R}^d)$, then $\varphi(f_n) \to \varphi(f)$ everywhere on $S'(\mathbb{R}^d)$, and thus also in probability and by Theorem 1.4 in L^2. Hence φ is a continuous map of $S(\mathbb{R}^d)$ into H and $\mathcal{E}$ is continuous.

If we, for simplicity, assume that $\mathcal{E}$ is non-degenerate, then $(S(\mathbb{R}^d), \mathcal{E})$ is a pre-Hilbert space and may be completed to a Hilbert space $S_{\mathcal{E}}$; φ then extends to an isometry of $S_{\mathcal{E}}$ onto the Gaussian Hilbert space H.

We have here regarded μ as given, but we may as well start from the bilinear form $\mathcal{E}$ instead, since it follows from Minlos's theorem (Gelfand and Vilenkin 1961, Chapter IV) that any continuous semi-definite symmetric bilinear form on $S(\mathbb{R}^d)$ corresponds to a (unique) Gaussian measure on $S'(\mathbb{R}^d)$ as above.

The most important special case is obtained by taking $\mathcal{E}(f,g) = \int fg\,dx$, the usual inner product in $L^2_{\mathbb{R}}(\mathbb{R}^d)$. Then the completion $S_{\mathcal{E}}$ equals $L^2_{\mathbb{R}}(\mathbb{R}^d)$, and the construction above yields a Gaussian measure μ on $S'(\mathbb{R}^d)$ and a Gaussian Hilbert space $H \subset L^2(\mu)$ such that $\varphi\colon L^2_{\mathbb{R}}(\mathbb{R}^d) \to H$ is an isometry. This Gaussian measure μ is called the *white noise measure* on $S'(\mathbb{R}^d)$. The study of the white noise measure and the corresponding Gaussian Hilbert space has developed into a whole theory, the *white noise calculus*, see Hida, Kuo, Potthoff and Streit (1993), Kuo (1996) and Obata (1994).

Another special case, known as the *free Euclidean Bose field in d dimensions* (Simon 1974, Hida, Kuo, Potthoff and Streit 1993, Glimm and Jaffe 1981, Chapter 6), is obtained by taking $\mathcal{E}(f,g) = \int f(1-\Delta)^{-1}g = (2\pi)^{-d}\int(1+|y|^2)^{-1}\hat{f}(y)\hat{g}(y)dy$, with $\hat{f}(y) = \int e^{-ix\cdot y} f(x)\,dx$. In this case $S_{\mathcal{E}}$ is the Sobolev space $\mathcal{H}_{-1}(\mathbb{R}^d)$.

EXAMPLE 1.17. Let us finally note that a closed subspace of a Gaussian Hilbert space is always a Gaussian Hilbert space. (See Example 1.11 for an important instance.)

We also introduce two further, closely related, definitions. Recall that a *linear isometry* of a Hilbert space into another is a linear map that preserves the norm, or, equivalently, the inner product. Linear isometries that are onto are also called *isomorphisms* (but note that this term has a different meaning for Banach spaces) or (in particular for complex spaces) *unitary* operators.

DEFINITION 1.18. A *Gaussian Hilbert space indexed by a (real) Hilbert space H* is a Gaussian Hilbert space G together with a specific linear isometry $h \mapsto \xi_h$ of H onto G.

On a formal level, this definition adds rather little, and it will not be much used in this book where the emphasis is on the Gaussian space itself. But this definition is important in contexts where the primary object of study is some (non-Gaussian) Hilbert space H, and the Gaussian space plays a secondary role.

DEFINITION 1.19. A *Gaussian field on a (real) Hilbert space H* is a linear isometry $h \mapsto \xi_h$ of H into some Gaussian space.

We do not require the isometry to be onto, but evidently its range $\{\xi_h : h \in H\}$ is a Gaussian Hilbert space indexed by H. Conversely, a Gaussian Hilbert space indexed by a Hilbert space H defines a Gaussian field on H.

EXAMPLE 1.20. We may now reinterpret Example 1.10 as an example of a Gaussian Hilbert space indexed by $L^2_{\mathbb{R}}([0,\infty), dt)$, with the defining isometry $f \mapsto \xi_f = \int_0^\infty f(t)\, dB_t$.

Alternatively, this isometry defines a Gaussian field on $L^2_{\mathbb{R}}([0,\infty), dt)$.

Analogously, any Gaussian field on an arbitrary L^2-space $L^2_{\mathbb{R}}(M,\mu)$ may be regarded as a stochastic integral; this will be studied in Section 7.2.

EXAMPLE 1.21. Example 1.16 gives a Gaussian Hilbert space indexed by the Hilbert space $S_{\mathcal{E}}$, and in the particular case of white noise a Gaussian Hilbert space indexed by $L^2_{\mathbb{R}}(\mathbb{R}^d)$. By the remark at the end of the preceding example, the white noise measure thus defines a stochastic integral for $L^2(\mathbb{R}^d)$. We will return to this stochastic integral in Section 7.2.

EXAMPLE 1.22. If H is a real Hilbert space, let $\{e_\alpha\}_{\alpha\in A}$ be an orthonormal basis in H. Let $\{\xi_\alpha\}_{\alpha\in A}$ be a set of independent standard normal variables with the same index set and define a Gaussian Hilbert space G as in Example 1.9. The mapping $\sum a_\alpha e_\alpha \mapsto \sum a_\alpha \xi_\alpha$ is an isometry of H onto G, and thus makes G a Gaussian Hilbert space indexed by H.

Example 1.22 provides one proof of the following fundamental result.

THEOREM 1.23. *If H is a real Hilbert space, then there exists a Gaussian Hilbert space indexed by H, and thus a Gaussian field on H.* □

We continue with more examples. Except the first, they are rather technical, and although they describe important constructions, they may be somewhat misleading here. The central idea in this book is to study Gaussian spaces in general, without caring about how they are constructed. (For our purposes, we can always use the simple construction in Example 1.22.) We may therefore ignore all technical problems concerning for example the existence of Gaussian measures in vector spaces. (Of course, for other purposes it is important to study constructions such as these in detail; see for example the references given below.) In our viewpoint, Gaussian spaces are simple objects by themselves, although they sometimes have complicated constructions and relations to other objects.

EXAMPLE 1.24. If H is a finite-dimensional real Hilbert space, let ξ be an H-valued Gaussian random variable such that $\langle \xi, h\rangle \sim \mathrm{N}(0, \|h\|^2)$ for every $h \in H$. Then $h \mapsto \langle \xi, h\rangle$ defines a Gaussian Hilbert space indexed by H. (To construct such a ξ, choose an orthonormal basis $e_1, \dots, e_n$ in H and let $\xi = \sum_1^n \xi_i e_i$ with $\xi_1, \dots, \xi_n$ independent standard normal. Alternatively, we may identify H with $\mathbb{R}^d$ and let ξ have the standard Gaussian distribution defined in Example 1.14.)

EXAMPLE 1.25. (Rather technical, and may be skipped.) If H is a real Hilbert space of infinite dimension, the construction of Example 1.24 does not work, because no such ξ exists. (Equivalently, in an infinite-dimensional Hilbert space, there is no 'standard Gaussian measure' similar to the one constructed for $\mathbb{R}^d$ in Example 1.14.) Nevertheless, we may find ξ as a random variable in a suitable larger space.

Take for example $H = \ell^2$ and define $\xi = (\xi_1, \xi_2, \dots)$, where the ξ_k are independent standard normal variables. Then a.s. $\xi \notin \ell^2$, but ξ is a well-defined random variable in a larger space, for example the Hilbert space $\{(x_k)_1^\infty : \sum_1^\infty (x_k/k)^2 < \infty\}$ or the product space $\mathbb{R}^\infty$. Moreover, for any $a = (a_k)_1^\infty \in \ell^2$, the inner product $\langle \xi, a\rangle = \sum_1^\infty a_k \xi_k$ is well-defined a.s., and the mapping $a \mapsto \langle \xi, a\rangle$ defines a Gaussian Hilbert space indexed by ℓ^2. (Note that the sum $\sum_1^\infty a_k \xi_k$ converges a.s. for every $a \in \ell^2$, but the exceptional null set depends on a and the union of these null sets is almost the whole probability space because $\xi \notin \ell^2$ a.s.)

The general situation is this. Let $\mathcal{X}$ be a real locally convex topological vector space which contains the Hilbert space H as a dense subspace, with a continuous inclusion mapping $H \to \mathcal{X}$. Then there is a dual inclusion $\mathcal{X}^* \subset H^* \cong H$, so we have a triple $\mathcal{X}^* \subset H \subset \mathcal{X}$. Under suitable topological conditions, there exists a Gaussian random variable ξ in $\mathcal{X}$ such that $\langle \xi, \varphi\rangle \sim \mathrm{N}(0, \|\varphi\|_H^2)$ for every $\varphi \in \mathcal{X}^*$. (Equivalently, there exists a Gaussian measure on $\mathcal{X}$ with the right finite-dimensional marginals.) This holds for example (Gelfand and Vilenkin 1961, Chapter IV) if $\mathcal{X}$ is another Hilbert space such that the inclusion mapping $H \to \mathcal{X}$ is a Hilbert–Schmidt operator, or if $\mathcal{X}$ is the dual of a nuclear space $\mathcal{Y} \subset H$ which is dense in H. (See e.g. Obata (1994), Schaefer (1971) or Treves (1967) for the definition of nuclear spaces.) Examples of such nuclear spaces $\mathcal{Y}$ are the direct sum $\sum_1^\infty \mathbb{R}$, in which case the dual space $\mathcal{X}$ is the the product space $\mathbb{R}^\infty$ mentioned above, and the space $S(\mathbb{R}^d)$ in Example 1.16, in which case $\mathcal{X}$ is the dual space of tempered distributions; in these cases $\mathcal{X}^* = \mathcal{Y}$ because the spaces are reflexive.

Another case where such a Gaussian variable ξ exists is when $\mathcal{X}$ is a Banach space with a norm satisfying a certain tightness (or continuity) condition, see Gross (1967) and Kuo (1975). (Such a space $\mathcal{X}$ is often called an *abstract Wiener space.*)

Suppose that such a variable ξ exists. Then the mapping $\varphi \mapsto \langle \xi, \varphi\rangle_{\mathcal{X}}$ is an isometry of the dense subspace $\mathcal{X}^*$ of H onto the Gaussian linear space