1

The Abstract Background

1.1 Introduction

This chapter is devoted to the general functional-analytic preliminaries which will be needed throughout the book. A good deal of what is presented may be familiar to the reader in a Banach space setting, but in view of later applications we give the theory in the more general context of quasi-Banach spaces, omitting details when they are too similar to those in Banach spaces. In particular, we present a Riesz theory for compact linear operators acting in a quasi-Banach space, detailing the nature of the spectrum.

The bulk of the chapter concerns the basic properties of entropy and approximation numbers of bounded linear operators. Of especial importance is the connection between eigenvalues and entropy numbers which is established in 1.3.4. In a Banach space, this was proved by Carl [Carl1]; the quasi-Banach version given here extends the method in [CaT]. This connection plays a fundamental role in the estimates of eigenvalues of (degenerate) elliptic (pseudo)differential operators which are given in Chapter 5.

1.2 Spectral theory in quasi-Banach spaces

We begin by recalling some basic facts about quasi-normed spaces. A *quasi-norm* on a complex linear space B is a map $\|\cdot\,|B\|$ from B to the non-negative reals $\mathbf{R}_+$ such that

(i) $\|x\,|B\| = 0$ if, and only if, $x = 0$,
(ii) $\|\lambda x\,|B\| = |\lambda|\,\|x\,|B\|$ for all scalars $\lambda \in \mathbf{C}$ and all $x \in B$,
(iii) there is a constant C such that for all $x, y \in B$,

$$\|x + y\,|B\| \le C\left(\|x\,|B\| + \|y\,|B\|\right).$$

Plainly $C \geq 1$; if $C = 1$ is allowed then $\|\cdot|B\|$ is a norm on B. Each quasi-norm defines a topology on B, compatible with the linear structure of B and with a basis of (not necessarily open) neighbourhoods of any point $x \in B$ given by the sets $\{y \in B : \|x - y|B\| < 1/n\}$, $(n \in \mathbf{N})$. The pair $(B, \|\cdot|B\|)$ is called a *quasi-normed space*; it is a particular type of (metrisable) topological vector space. Cauchy sequences in a quasi-normed space B are defined in the obvious way; if every Cauchy sequence in B converges (to a point of B), we call B a *quasi-Banach* space.

Given any $p \in (0, 1]$, a *p-norm* on a linear space B is a map $\|\cdot|B\| \to \mathbf{R}_+$ which satisfies conditions (i) and (ii) above and instead of (iii) satisfies

(iii′) $\|x + y|B\|^p \leq \|x|B\|^p + \|y|B\|^p$ for all $x, y \in B$.

Two quasi-norms or p-norms $\|\cdot|B\|_1$ and $\|\cdot|B\|_2$ are said to be *equivalent* if there is a constant $c \geq 1$ such that for all $x \in B$,

$$c^{-1}\|x|B\|_1 \leq \|x|B\|_2 \leq c\,\|x|B\|_1 .$$

It can be shown that (see [Kön], p. 47 or [DeVL], p. 20) if $\|\cdot|B\|_1$ is a quasi-norm on B, then there exist $p \in (0, 1]$ and a p-norm $\|\cdot|B\|_2$ on B which is equivalent to $\|\cdot|B\|_1$; the connection between p and the constant C which appears in (iii) above is that C can be taken to be $2^{\frac{1}{p}-1}$. Conversely, any p-norm is a quasi-norm with $C = 2^{\frac{1}{p}-1}$.

Let $0 < q \leq \infty$ and let ℓ_q be the set of all complex sequences $b = (b_k)_{k \in \mathbf{N}}$ of scalars such that

$$\|b|\ell_q\| := \begin{cases} \left(\sum_{k=1}^{\infty} |b_k|^q\right)^{1/q}, & q < \infty, \\ \sup_{k \in \mathbf{N}} |b_k|, & q = \infty, \end{cases}$$

is finite. Then it is easy to see that ℓ_q is a quasi-Banach space, and even a Banach space if $q \geq 1$. In the same way, it can be verified that the L_q spaces are quasi-Banach spaces, and even Banach spaces if $q \geq 1$, when endowed with the obvious (quasi-)norm.

Let A, B be quasi-Banach spaces and let $T : A \to B$ be linear. Just as for the Banach space case, T will be called *bounded* or *continuous* if

$$\|T\| := \sup\{\|Ta|B\| : a \in A, \|a|A\| \leq 1\} < \infty.$$

The family of all such T will be denoted by $L(A, B)$, or $L(A)$ if $A = B$. For the most part, terminology which is standard in the context of Banach spaces will be taken over without special comment to the quasi-Banach situation. We shall, however, be explicit about certain spectral matters to which we now turn.

Let B be a (complex) quasi-Banach space and let $\mathscr{T}$ be the family of all closed, linear, densely defined operators in B, so that any $T \in \mathscr{T}$ has domain dom(T) which is a dense linear subspace of B mapped into B by T. Given any $T \in \mathscr{T}$, the *resolvent set* of T is

$$\rho(T) = \left\{\lambda \in \mathbf{C} : (T - \lambda \mathrm{id})^{-1} \text{ exists and belongs to } L(B)\right\}.$$

Here id stands for the identity map of B to itself. The *spectrum* of T is $\sigma(T) = \mathbf{C} \backslash \rho(T)$, and we distinguish two subsets of it, the *point spectrum* $\sigma_p(T)$ and the *essential spectrum* $\sigma_e(T)$. By $\sigma_p(T)$ we simply mean the set of all eigenvalues of T, so that $\lambda \in \sigma_p(T)$ if, and only if, $\lambda \in \mathbf{C}$ and there exists $u \in B\backslash\{0\}$ such that $Tu = \lambda u$. We choose to define $\sigma_e(T)$ by means of *Weyl sequences* (sometimes called *singular sequences*): a sequence $\{u_j\}$ in dom(T) is called a Weyl sequence of T corresponding to $\lambda \in \mathbf{C}$ if it does not contain a convergent subsequence and $\|u_j\| = 1$, $j \in \mathbf{N}$, $Tu_j - \lambda u_j \to 0$ as $j \to \infty$. Now we can define

$$\sigma_e(T) = \{\lambda \in \mathbf{C} : \text{there is a Weyl sequence of } T \text{ corresponding to } \lambda\}.$$

For information about the essential spectrum in a Banach space context we refer to [EEv], especially Chapter IX.

If $T \in \mathscr{T}$ and $\lambda \in \rho(T)$, then of course there exists $c > 0$ such that

$$\|Tu - \lambda u\| \geq c\,\|u\| \quad \text{for all } u \in \mathrm{dom}(T). \tag{1}$$

By analogy with self-adjoint or normal operators in a Hilbert space, we introduce a class of operators characterised by property (1). We define

$$\mathscr{S} = \{T \in \mathscr{T} : \lambda \in \rho(T) \text{ if, and only if, (1) holds for some } c > 0\}.$$

With operators having pure point spectrum in mind (see [Tri1], 4.5.1, p. 254) we define

$$\mathscr{S}_0 = \{T \in \mathscr{S} : \sigma_e(T) = \emptyset\}. \tag{2}$$

Two further classes of operators will be useful:

$$\mathscr{K} = \{T \in L(B) : T \text{ is compact}\} \tag{3}$$

and

$$\mathscr{S}_1 = \left\{T \in \mathscr{T} : \rho(T) \neq \emptyset,\ (T - \lambda \mathrm{id})^{-1} \in \mathscr{K} \text{ for some } \lambda \in \rho(T)\right\}. \tag{4}$$

Proposition 1 *Let $T \in \mathscr{T}$. Then the following three assertions are equivalent:*

(i) $T \in \mathscr{S}$;

(ii) $\sigma(T) = \sigma_e(T) \cup \sigma_p(T)$;
(iii) $\sigma(T) = \{\lambda \in \mathbf{C}$: there exists a sequence $\{u_j\}$ in dom(T) with $\|u_j\| = 1$, $j \in \mathbf{N}$, and $Tu_j - \lambda u_j \to 0$ as $j \to \infty\}$.

Proof (i) $\Longleftrightarrow$(iii). This follows immediately from (1).

(ii) $\Longleftrightarrow$ (iii). Let M be the right-hand side of (iii): we have to prove that $\sigma_e(T) \cup \sigma_p(T) = M$. Let $\lambda \in M$. If there is a sequence $\{u_j\}$ in dom(T) with no convergent subsequence, and with $\|u_j\| = 1$ and $Tu_j - \lambda u_j \to 0$, then by definition, $\lambda \in \sigma_e(T)$. On the other hand, if there is a sequence $\{u_j\}$ with the same properties save that it is convergent, to u say, then $\|u\| = 1$ and $Tu_j \to \lambda u$. Since T is closed it follows that $u \in$ dom(T) and $Tu = \lambda u$, so that $\lambda \in \sigma_p(T)$. Hence $M \subset \sigma_e(T) \cup \sigma_p(T)$. As the reverse inclusion is obvious, the proof is complete.

Remark 1 The class $\mathscr{S}$ is based on self-adjoint operators in a Hilbert space, which have empty residual spectrum (see [EEv], pp. 5, 414–15; [Tri1], 4.2.3, pp. 219–21). For corresponding assertions concerning normal operators in Hilbert spaces, see [Rud], pp. 312–313.

We recall that the *geometric multiplicity* of an eigenvalue λ of T is $\dim \ker(T - \lambda \mathrm{id})$; its *algebraic multiplicity* is $\dim \bigcup_{n=1}^{\infty} \ker(T - \lambda \mathrm{id})^n$.

Proposition 2 *(i) If* $T \in \mathscr{S}_0$, *then*

$$\sigma(T) = \{\lambda \in \sigma_p(T) : \lambda \text{ has finite geometric multiplicity}\}.$$

(ii) If $T \in \mathscr{S}_1$, *then* $(T - \lambda \mathrm{id})^{-1} \in \mathscr{K}$ *for all* $\lambda \in \rho(T)$.

Proof (i) In view of (2) and Proposition 1, we merely have to exclude eigenvalues of infinite geometric multiplicity. Suppose that $\lambda \in \sigma_p(T)$ is such that $\dim \ker(T - \lambda \mathrm{id}) = \infty$. Then given any $\varepsilon \in (0,1)$, there is a sequence $\{u_j\}$ in $\ker(T - \lambda id)$ such that $\|u_j\| = 1$ for each j and $\|u_j - u_k\| > 1 - \varepsilon$ for $j \neq k$: this follows by using the same proof as in the Banach space case (see [Tri1], pp. 23, 24). Hence $\lambda \in \sigma_e(T)$ and we have a contradiction.

(ii) That $(T - \lambda \mathrm{id})^{-1} \in \mathscr{K}$ for some $\lambda \in \rho(T)$ if, and only if, $(T - \mu \mathrm{id})^{-1} \in \mathscr{K}$ for all $\mu \in \rho(T)$ follows immediately from the resolvent equation

$$T_\lambda = T_\mu + (\lambda - \mu) T_\lambda T_\mu, \qquad T_\lambda = (T - \lambda \mathrm{id})^{-1}, \; T_\mu = (T - \mu \mathrm{id})^{-1},$$

which holds just as in the Banach space case.

Remark 2 As in the case of self-adjoint operators acting in a Hilbert space (see [Tri1], 4.2.3 and 4.2.4), eigenvalues of infinite geometric multiplicity belong to both $\sigma_p(T)$ and $\sigma_e(T)$.

Remark 3 Proposition 2(ii) simply shows that the definition of $T \in \mathscr{S}_1$ is independent of the particular $\lambda \in \rho(T)$.

Remark 4 If B is a Banach space and $T \in \mathscr{S}_1$, then (see [EEv], Theorem IX.3.1, p. 423)

$$\sigma(T) = \{\lambda \in \sigma_p(T) : \lambda \text{ has finite algebraic multiplicity}\},$$

and for all $\lambda \in \rho(T)$, $(T - \lambda \mathrm{id})^{-1} \in \mathscr{K}$ is a Riesz operator. (For Riesz operators we refer to [Pi1], Chapters 26 & 27 and to [Kön], p. 18.) Hence $\mathscr{S}_1 \subset \mathscr{S}_0$ in the Banach space case.

We now aim to develop a Riesz theory for compact operators, which amounts to extending Remark 4 to quasi-Banach spaces.

Theorem *Let B be a (complex) infinite-dimensional quasi-Banach space and let $K \in \mathscr{K}$.*

(i) $K \in \mathscr{S}$,
(ii) $\sigma_e(K) = \{0\}$, $\sigma(K) = \{0\} \cup \sigma_p(K)$,
(iii) $\sigma(K) \setminus \{0\}$ *consists of an at most countably infinite number of eigenvalues of finite algebraic multiplicity which may accumulate only at the origin.*

Proof We proceed in several steps.

Step 1 Let $\lambda \in \sigma_e(K)$, assume that $\lambda \neq 0$, and let $\{u_j\}$ be a related Weyl sequence. Then since K is compact, we may suppose that $\{Ku_j\}$ is convergent, and as

$$\lambda u_j = \left(\lambda u_j - Ku_j\right) + Ku_j,$$

we see that $\{u_j\}$ must be convergent. This contradiction shows that $\sigma_e(K) \subset \{0\}$.

Step 2 To prove that $0 \in \sigma_e(K)$, let $\varepsilon \in (0,1)$. Let $\{u_j\}_{j \in \mathbf{N}}$ be a sequence in B such that for all $j \in \mathbf{N}$, $\|u_j\| = 1$ and

$$\operatorname{dist}\left(u_j, \operatorname{span}(u_1, ..., u_{j-1})\right) > 1 - \varepsilon, \quad j > 1.$$

The existence of such a sequence follows just as in the Banach space case; see [Tri1], pp. 23, 24. Put $v_j = u_{2j} - u_{2j-1}$, $j \in \mathbf{N}$. Then for each $j \in \mathbf{N}$, $1 - \varepsilon \le \|v_j\| \le C$; and if $k < j$, since $u_{2j-1} + u_{2k} - u_{2k-1} \in \operatorname{span}(u_1, ..., u_{2j-1})$, we have

$$\|v_j - v_k\| = \|u_{2j} - u_{2j-1} - u_{2k} + u_{2k-1}\| > 1 - \varepsilon.$$

Since K is compact we may suppose that $\{Ku_j\}$ is convergent; thus $Kv_j \to 0$ as $j \to \infty$. Hence $\{v_j / \|v_j\|\}$ is a Weyl sequence corresponding to 0, and so $0 \in \sigma_e(K)$. This proves the first part of (ii).

Step 3 Let $\lambda \in \sigma_p(K)$, $\lambda \ne 0$. Then λ has finite algebraic multiplicity: this follows from the Banach space arguments given in [EEv], Lemma I.1.18, p. 9 with obvious changes. In the same way, natural adaptations of the proof of [Ka], Theorem III.6.26, p. 185 show that $\sigma_p(K)$ has no accumulation point different from zero and is, of course, countable.

Step 4 First renorm B with an equivalent p-norm. We claim that if $\mu \in \mathbf{C}$, $|\mu| > \|K\|\, C$ for a suitable positive C, then $\mu \in \rho(K)$. To establish this, the Banach space arguments, involving Neumann series, from [Tri1], pp. 67–70, can be applied, working with the pth power of this p-norm rather than the p-norm itself.

Step 5 Let $\lambda \in \mathbf{C}$, $\lambda \ne 0$, $\lambda \notin \sigma_p(K)$. By Step 1, $\lambda \notin \sigma_e(K)$ and so there exists $c > 0$ such that

$$\|Ku - \lambda u\| \ge c\,\|u\| \text{ for all } u \in B. \tag{5}$$

Now let $\mu \in \mathbf{C}$ be such that $|\mu| > \|K\|\, C$ in the sense of Step 4, and let γ be a path in $\mathbf{C}$ joining μ to λ which does not pass through any eigenvalues. A simple contradiction argument shows that (5) holds with λ replaced by any κ on γ and with the same constant c for all κ. Let $\kappa \in \rho(K)$. Then

$$\left\|(K - \kappa \mathrm{id})^{-1}\right\| \le c^{-1}.$$

Let ν be near κ on γ. Then for all $u \in B$,

$$v = Ku - \nu u = Ku - \kappa u - (\nu - \kappa)u \tag{6}$$

and hence

$$(K - \kappa \mathrm{id})^{-1} v = u - (\nu - \kappa)(K - \kappa \mathrm{id})^{-1} u. \tag{7}$$

We arrange that $|\nu - \kappa|$ should be so small that $|\nu - \kappa| \left\|(K - \kappa I)^{-1}\right\| < 1$. Thus by the Neumann series argument, (7), and so also (6), has a unique solution. Hence by (5) with λ replaced by ν, it follows that $\nu \in \rho(K)$. Iteration of this shows that $\lambda \in \rho(K)$. The proof of the theorem is complete.

Proposition 3 $\mathscr{S}_1 \subset \mathscr{S}_0$.

Proof Let $\lambda = 0$ in (4).The result follows from the theorem and the fact that $0 \neq \mu \in \sigma_e(T)$ if, and only if, $\mu^{-1} \in \sigma_e(T^{-1})$.

1.3 Entropy numbers and approximation numbers

1.3.1 Definitions and elementary properties

We begin with entropy numbers.

Definition 1 *Let A, B be quasi-Banach spaces and let $T \in L(A, B)$; put $U_A = \{a \in A : \|a\,|A\| \le 1\}$. Then for all $k \in \mathbf{N}$, the* kth *entropy number $e_k(T)$ of T is defined by*

$$e_k(T) = \inf\left\{\varepsilon > 0 : T(U_A) \subset \bigcup_{j=1}^{2^{k-1}} \left(b_j + \varepsilon U_B\right) \text{ for some } b_1, ..., b_{2^{k-1}} \in B\right\}.$$

Remark 1 This definition has its roots in the notion of the metric entropy of a set which Kolmogorov introduced in the 1930s: given any compact subset K of a metric space and any $\varepsilon > 0$, Kolmogorov denoted by $N_\varepsilon(K)$ the least $N \in \mathbf{N}$ such that K can be covered by N balls of radius ε, and called $H_\varepsilon(K) := \log_2 N_\varepsilon(K)$ the metric entropy of K. If A, B are Banach spaces and $T \in L(A, B)$ is compact, then the entropy numbers of T are obtained, roughly speaking, by solving the equation $H_\varepsilon(T(U_A)) = k - 1$ for $\varepsilon = e_k(T)$.

The following lemma gives some elementary properties of entropy numbers.

Lemma 1 *Let A, B, C be quasi-Banach spaces, let $S, T \in L(A, B)$ and suppose that $R \in L(B, C)$.*

(i) $\|T\| \ge e_1(T) \ge e_2(T) \ge \cdots \ge 0$; $e_1(T) = \|T\|$ *if B is a Banach space.*

(ii) *For all* $k, \ell \in \mathbf{N}$,

$$e_{k+\ell-1}(R \circ S) \le e_k(R) e_\ell(S).$$

(iii) *If B is a p-Banach space* $(0 < p \le 1)$, *then for all* $k, \ell \in \mathbf{N}$,

$$e^p_{k+\ell-1}(S+T) \le e^p_k(S) + e^p_\ell(T).$$

Proof (i) Since $\|T\| = \inf\{\mu \ge 0 : T(U_A) \subset \mu U_B\}$, it is plain that $e_1(T) \le \|T\|$. If $T(U_A) \subset b_0 + \mu U_B$ for some $b_0 \in B$ and some $\mu \ge 0$, then given any $a \in U_A$, there are elements $b_1, b_2 \in U_B$ such that $Ta = b_0 + \mu b_1$ and $-Ta = b_0 + \mu b_2$. Hence $2Ta = \mu(b_1 - b_2)$ and $\|Ta\,|B\| \le \mu 2^{(1/p)-1}$ for a $p \in (0,1]$ such that B is a p Banach space. It follows immediately that if B is a Banach space, then $e_1(T) = \|T\|$. The rest of (i) is obvious.

(iii) Let $\mu > e_k(S)$ and $\nu > e_\ell(T)$. Then there are points $b_1, ..., b_M$, $b'_1, ..., b'_N \in B$ such that

$$S(U_A) \subset \bigcup_{i=1}^{M} (b_i + \mu U_B), \ T(U_A) \subset \bigcup_{j=1}^{N} (b'_j + \nu U_B),$$

where $M \le 2^{k-1}$ and $N \le 2^{\ell-1}$. Given any $a \in U_A$, there are points b_i, b'_j such that $Sa \in b_i + \mu U_B$, $Ta \in b'_j + \nu U_B$. Hence

$$(S+T)a \in b_i + b'_j + \mu U_B + \nu U_B \subset b_i + b'_j + (\mu^p + \nu^p)^{1/p}\, U_B,$$

and thus

$$(S+T)U_A \subset \bigcup_{i=1}^{M} \bigcup_{j=1}^{N} \left\{ b_i + b'_j + (\mu^p + \nu^p)^{1/p}\, U_B \right\}.$$

The number of points $b_i + b'_j$ with $i \in \{1, ..., M\}$, $j \in \{1, ..., N\}$ is at most $MN \le 2^{(k+\ell-1)-1}$. The result follows.

(ii) The proof of this is similar to that of (iii), and is left to the reader.

Remark 2 Since the $e_k(T)$ decrease as k increases, and are non-negative, $\lim_{k\to\infty} e_k(T)$ exists and plainly equals

$$\inf\{\varepsilon > 0 : T(U_A) \text{ can be covered by finitely many } B\text{-balls of radius } \varepsilon\}.$$

Following the terminology used in the Banach space case (see [EEv], II.1, p. 47), we shall refer to this limit as the *ball measure of non-compactness* of T and denote it by $\widetilde{\beta}$ (T); plainly $\widetilde{\beta}(T) = 0$ if, and only if, T is compact.

Remark 3 Unless $T \in L(A, B)$ is the zero operator, it is clear that for all $k \in \mathbf{N}$, $e_k(T) \neq 0$. The dependence of $e_k(T)$ upon k is, of course, a function of the particular operator T, and we shall devote considerable space to the determination of this dependence for cases of special importance. For the moment we merely note that if A is a (complex) Banach space of dimension $m < \infty$ and id: $A \to A$ is the identity map, then (see [EEv], II.1, p. 49) for all $k \in \mathbf{N}$,

$$1 \leq 2^{(k-1)/2m} e_k(\mathrm{id}) \leq 4.$$

Moreover, if A and B are (real or complex) Banach spaces and $T \in L(A, B)$, $T \neq 0$ has the property that there are positive numbers c and ρ such that for all $k \in \mathbf{N}$, $e_k(T) \leq c 2^{-\rho k}$, then it turns out that T is of finite rank (see [CaS], 1.3, especially pp. 20–21).

Remark 4 Let A be a quasi-Banach space with a non-convex unit ball U_A (ℓ_p, with $0 < p < 1$, is such a space). Then it may happen that there exists $a \in U_A$ such that

$$a + \{b : b = \lambda b_0, \|b_0\| = 1, \ \lambda \in [-1, 1]\} \subset \mu U_A$$

for some $\mu \in (0, 1)$. In that case it is easy to find an operator $T \in L(A)$ with $e_1(T) < \|T\|$.

The relationship between the entropy numbers of a map $T \in L(A, B)$, A and B being Banach spaces, and those of its adjoint T^* has attracted a good deal of interest (see Carl [Carl3], Edmunds and Tylli [ETy], Gordon, König and Schütt [GKS] and Bourgain et al [BPST]) and is still not completely resolved. However, when A and B are Hilbert spaces the question is settled by the following result:

Theorem *Let $T \in L(H_1, H_2)$, where H_1 and H_2 are Hilbert spaces. Then for all $k \in \mathbf{N}$, $e_k(T) = e_k(T^*) = e_k\left(|T|\right)$, where $|T|$ is the positive square root of T.*

Proof By the Polar Decomposition Theorem (see [EEv], IV.3, p. 180) there is a partial isometry $U \in L(H_1, H_2)$ from $(\ker T)^\perp$ to $\overline{\mathrm{im} T}$ such that $T = U|T|$ and $|T| = U^* T$. It follows that for all $k \in \mathbf{N}$,

$$e_k(T) \leq e_k\left(|T|\right) \|U\| = e_k\left(|T|\right)$$

and

$$e_k\left(|T|\right) \leq e_k(T) \|U^*\| = e_k(T).$$

Hence $e_k(T) = e_k(|T|)$. Use of the facts that $T^* = |T|\,U^*$ and $|T| = T^*U$ (see [EEv], Theorem IV.3.2) leads to $e_k(T^*) = e_k(|T|)$, and the proof is complete.

Remark 5 Outside of Hilbert spaces the results of Bourgain et al [BPST] will be of particular interest to us.To explain these we need to introduce some terminology. Given any bounded sequence $x = \{x_n\}_{n\in\mathbf{N}}$ of scalars, put

$$x_n^* = \inf\{\sigma \geq 0 : \#\{k \in \mathbf{N} : |x_k| \geq \sigma\} < n\}, \qquad n \in \mathbf{N};$$

the sequence $\{x_n^*\}_{n\in\mathbf{N}}$ is called the *non-increasing rearrangement* of x. A Banach space $(X, \|\cdot|X\|)$ consisting of sequences $\{x_n\}_{n\in\mathbf{N}}$ of scalars is called a *symmetric sequence space* if $\|\{x_n\}|X\| = \|\{x_n^*\}|X\|$ for all $\{x_n\} \in X$; and in that case, $\|\cdot|X\|$ is called a *symmetric norm*. Particular examples of symmetric spaces are the Lorentz spaces: given any $p, q \in (1, \infty]$, the Lorentz space $\ell_{p,q}$ is the linear space of all $x = \{x_n\}_{n\in\mathbf{N}} \in \ell_\infty$ such that

$$\|x|\ell_{p,q}\| = \begin{cases} \left(\sum_{n=1}^{\infty}\left(n^{\frac{1}{p}-\frac{1}{q}}x_n^*\right)^q\right)^{1/q} & \text{if} \quad 1 < q < \infty, \\ \sup_{n\in\mathbf{N}}\left(n^{1/p}x_n^*\right) & \text{if} \quad q = \infty, \end{cases}$$

is finite.

The result of [BPST] is as follows:

Let A be a uniformly convex Banach space, let B be a Banach space and let $T \in L(A, B)$ be compact.

(i) There is a positive constant c, depending only on A, such that for all $m \in \mathbf{N}$ and all $p \in [1, \infty)$,

$$c^{-1}\left(\sum_{k=1}^{m} e_k^p(T^*)\right)^{1/p} \leq \left(\sum_{k=1}^{m} e_k^p(T)\right)^{1/p} \leq c\left(\sum_{k=1}^{m} e_k^p(T^*)\right)^{1/p}.$$

The same holds with the ℓ_p norm replaced by any symmetric norm.

(ii) If, for some $k \in \mathbf{N}$,

$$e_k(T) \leq Ce_{2k}(T),\; e_k(T^*) \leq Ce_{2k}(T^*),$$

then

$$C_1^{-1}e_k(T^*) \leq e_k(T) \leq C_1 e_k(T^*),$$

where C_1 depends only on A and C.

Note that (i) implies, by taking the $\ell_{p,\infty}$ norm, that if $e_k(T) = O(k^{-1/p})$ as $k \to \infty$, then $e_k(T^*) = O(k^{-1/p})$.