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Ian Richards and Heekyung Youn
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CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521371490

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First published 1990
First paperback edition 1995
Re-issued in this digitally printed version 2007

A catalogue record for this publication is available from the British Library

ISBN 978-0-521-37149-0 hardback
ISBN 978-0-521-55890-7 paperback

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PREFACE

Distributions are sometimes called 'generalized functions', and that is essentially what they are. They correspond to situations presented to us by physical experience which are not adequately covered by the traditional $y = f(x)$ notion of a function. An example is the well-known Dirac Delta Function, which is in fact not a function in the standard sense. The Dirac 'function' corresponds to a unit impulse imparted to a system over what we may idealize as an infinitely short interval of time. Think, for example, of an object being struck by a hammer. While in reality there is some compression of the hammer and of the object, and a small but finite time span during which the interaction occurs, that is not the way we normally see it. To the unaided eye, the whole thing takes place: Bang! – in an instant. This idealization not only corresponds to human intuition, but is very useful in physical applications.

Here an aside. In this discussion, when we use the term 'physical', we really mean 'phenomenological' – i.e. pertaining to the phenomena of nature. Thus, in our usage, the term physical could just as well apply to a problem in mathematical economics as to a problem in mechanics.

This still raises the question: Why create a whole theory to deal with an idea as simple as the Dirac Delta Function? Well, firstly, the idea may not be quite so simple as it looks. More importantly, the idea has important generalizations, each of which could be treated directly on its own merits, but only at the expense of an ever widening loss of clarity and comprehension. When the same idea, in different guises, recurs over and over, there should be an underlying theory which ties the different instances together. The notion of distributions, as introduced and codified by Laurent Schwartz, provides what is today the most widely used of such theories.

This book is addressed to non-specialists. It is intended to be somewhat in the spirit of Lighthill's splendid book **[LI]**. However, Lighthill uses a

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non-standard definition of 'distribution'. We believe that the standard approach is, in the long run, more powerful – and of course it has the merit of being standard. On the other hand, we have avoided the use of heavy theories, such as the theory of topological vector spaces. In the first five chapters of the book, we even get by without measure theory (although comments about the measure–theoretic connections occur from time to time).

In the same vein, we have taken great pains to motivate the developments and to make the proofs easy and clear. At times we might be accused of insulting the reader's intelligence. We do not feel that way at all. We assume that our reader is a busy person, who needs to know something about generalized functions, and who wants to learn the basic ideas with a minimum of pain and fuss. To whatever degree this book appears 'elementary', we have succeeded in our intention.

Of course, we do not claim that this book covers everything. There are many deeper results, which the reader can find in more advanced works, that lie outside the scope of ours. For one thing, the theory of topological vector spaces – which we have avoided – eventually comes in at more advanced stages of the theory.

We now give a brief description of the contents.

Chapter 1, as its title suggests, is introductory. It provides the motivation for Chapter 2, Part 1, which introduces the general class of Schwartz distributions. The reader who covers Chapter 2, Part 1, will already have the answer to the question: What is distribution theory about?

Chapter 2, Part 2, deals in a preliminary way with the convolution of distributions. This section is more difficult and could be omitted on first reading. Chapter 3 gives examples, and it could also be omitted on first reading. On the other hand, these examples are so much fun that it would seem rather a shame to ignore them.

Chapter 4 is intended as a preparation for Chapter 5. It gives the required background information on the Fourier transform in its classical setting. Besides merely presenting facts, we have taken pains in this chapter to present the underlying physical motivation.

Chapter 5 is (next to Chapter 2) the most important chapter in the book. It deals with the theory of 'tempered distributions'. Within this theory there is the splendid result that the Fourier transform of a tempered distribution is again a tempered distribution, and that the Fourier Inversion Theorem has universal validity. Anyone who has ever dealt with the Fourier transform in a classical setting – where the Fourier transform of one class of functions is usually a different class of functions, and special definitions are required even to make the theorems make sense – will appreciate the simplicity of the tempered distribution approach. Here again it is in the section on examples that the real fun occurs. One takes the Fourier transform of the

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function $f(x) = x$ (a function that is not even bounded, much less integrable), and discovers that the Fourier transform is a dipole (a generalized function that is not a function at all). Some of the examples, like the one just cited, are easy; others are harder. One example, starting with the rarified notion of 'generalized function', eventually boils down to a rather tricky computation of gamma function integrals. Finally, we must mention the beautiful result that a tempered distribution is periodic if and only if its Fourier transform is a sequence of delta functions. The coefficients of these delta functions form the Fourier series coefficients for the original distribution. Thus the theory of Fourier series is subsumed – and not by way of analogy but really subsumed – under the theory of the Fourier transform.

Chapter 6 gives the generalizations of distribution theory from the real line $\mathbb{R}^1$ to q-dimensional euclidean space $\mathbb{R}^q$. As noted there, this involves mainly a proliferation of subscripts. All of the basic ideas occur already for $\mathbb{R}^1$ (which is why we wrote most of the book for the one-dimensional case). We mention that a little measure theory is used in Chapter 6, in order to deal with space integrals. However, we have deliberately written the chapter so that an intuitive perception of space integrals should suffice.

Chapter 7 deals with the general problem of multiplication and convolution for distributions. While both multiplication and convolution were defined earlier (in Chapter 2), they were defined only subject to certain side conditions. By the way, these side conditions are absolutely standard – and for good reason: they make the theory easy! The general theory, as laid out in Chapter 7, is more difficult. So far as we know this problem has never been treated in textbook form, although there is a substantial research literature. The approach used here is based on earlier work by one of the authors (Youn) and represents an extension of her Ph.D. thesis.

In conclusion, we repeat a point made earlier. Much of the theory of distributions – the part that most non-specialists need to know – can be done without advanced methods. That is the main theme of this book.

Finally, we wish to thank Gian-Carlo Rota and our editor, David Tranah, for their guidance and patience. Thanks are also due to the staff of Cambridge University Press and the staff of the College of St Thomas for their support. Irene Pizzie of Cambridge University Press brought order to our occasionally chaotic manuscript. Mention must also be made of Susan Moro at St Thomas who did a spendid job of typing the book.

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