GROUP THEORY AND PHYSICS

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PREFACE

Group theory is one of the great achievements of 19th century mathematics. It emerged as a unifying idea drawing on four different sources: number theory, the theory of equations, geometry, and crystallography. The early motivation from number theory stemmed from the work of Euler, Legendre and Gauss on power residues. In the theory of equations, the study of various permutation groups became increasingly important through the work of Lagrange, Ruffini, Gauss, Abel, Cauchy, and especially Galois. The discovery of new types of geometries – including non-Euclidean, affine, projective etc. – led, eventually, to the famous Erlangen program of Klein, which proposed that the true study of any geometry lies in an analysis of its group of motions. In crystallography, the possible symmetries of the internal structure of a crystal were enumerated long before there was any possibility of its physical determination (by X-ray analysis).

The definition of an abstract group was proposed by Cayley in two remarkable papers in 1854, reflecting perhaps some influence of Boole (for abstract formulation) and Hamilton's quaternions (for the existence of algebras with noncommutative multiplication). This definition was not immediately appreciated by the mathematical community. In 1870, Kronecker (independently of Cayley) introduced the axioms for an abstract commutative group. When Cayley reiterated his definition in 1878, the reception was much warmer. In the period from 1870 to 1900, enormous progress was made in group theory. For example, the idea of a continuous group was introduced and studied by Lie; this led to a wealth of applications to geometry and differential equations, culminating in the classification by Killing and Cartan of the simple finite dimensional Lie groups. The theory of finite groups was greatly advanced through the work of Jordan, Hölder and Burnside.

The theory of group representations was created by Frobenius, Schur and Burnside in the last decade of the 19th century, although some of the ideas were anticipated in Jordan's monumental *Traité des substitutions* of 1870. Their theory for finite groups was extended to compact groups and brought into fruitful contact with Lie theory in a series of fundamental papers by Hermann Weyl in the 1920s. Almost all the key mathematical ideas presented in this book were developed during this period; thus, some 70 to 120 years ago. (The one principal exception is our description of Wigner's seminal paper, extending Frobenius's method to obtain the representations of the Poincaré group. This paper appeared around 50 years ago.) Of course, there has been huge progress in the last half century. I have tried to present this classical material from a geometric viewpoint, which will, hopefully, help the reader to enter the realm of the more recent advances.

It is more difficult to trace the early sources of the applications of group theory to physics. Symmetry considerations entered into the solutions of physical problems at the very beginning of mathematical physics. Mathematical crystallography, a major success of 19th century physics, is essentially group theoretical, but it had developed before the abstract language of group theory had been accepted. We explain some of the more elementary ideas of this subject in Chapter 1, and go into somewhat more detail in Appendix A. The spirit of Klein's Erlangen program pervades Poincaré's *La Science et l'Hypothese*, and other philosophical writings, and through them influenced the development of special relativity. The culmination of this group theoretical approach to relativity is Wigner's paper mentioned above, where the physical characteristics, mass and spin, arise as parameters in the description of irreducible representations. One of the goals of our method of presentation is to reach this central result.

The explicit recognition of the importance of group representation theory in physics started very soon after the discovery of quantum mechanics, with the path-breaking work of Weyl, Wigner, and others. In fact, Weyl's classic book of 1928, *Gruppentheorie und Quantenmechanik*, makes instructive and inspiring reading even today. (In his book, Weyl adopts the pedagogic strategy of segregating the mathematics and the physics into separate chapters. There is much to be said for this strategy, especially from the point of view of logical coherence. But it had the unintended effect that physicists and mathematicians would read alternate chapters. I have taken the risk of going to the opposite extreme here, trying to use the physics to motivate the mathematics and vice versa, mixing the two.) The uses of group theory in quantum mechanics extended from chemistry and spectroscopy in the 1920s and 1930s, to nuclear and particle physics in the 1930s and 1940s, and then to high energy physics and the discovery of the theory of colored quarks in the 1960s and 1970s. It is this story of the interweaving of mathematics and physics that I try to tell in this book.

It should not be supposed that there was a warm reception in the physics community to the introduction of group theoretical methods. In fact, the contrary was true. To get a feeling for a typical early reaction, let me quote at length from the autobiography of John Slater, who was a leading American physicist and head of the MIT Physics Department for many years. The following quotes are taken from pages 60–2 of his autobiography:

> It was at this point that Wigner, Hund, Heitler, and Weyl entered the picture with their "Gruppenpest": the pest of the group theory.... The authors of the "Gruppenpest" wrote papers which were incomprehensible to those like me who had not studied group theory, in which they applied these theoretical results to the study of the many electron problem. The practical consequences appeared to be negligible, but everyone felt that to be in the mainstream one had to learn about it. Yet there were no good texts from which one could learn group theory. It was a frustrating experience, worthy of the name of a pest.

> I had what I can only describe as a feeling of outrage at the turn which the subject had taken....
>
> As soon as this [Slater's] paper became known, it was obvious that a great many other physicists were as disgusted as I had been with the group-theoretical approach to the problem. As I heard later, there were remarks made such as "Slater has slain the 'Gruppenpest'". I believe that no other piece of work I have done was so universally popular.

Outrage, disgust, the characterization of group theory as a plague or as a dragon to be slain – this is not an atypical physicist's reaction in the 1930s–50s to the use of group theory in physics. It is, however, amazing to consider that this autobiography was published in 1975, after the major triumphs of group theory in elementary particle physics.

When I was a student in the early 1950s, the basic facts of abstract group theory were part of the algebra course, but the theory of group representations was not included in the standard mathematics curriculum. My introduction to representation theory and its physical applications was at the hands of Prof. George W. Mackey. He gave a wonderful and justly famous course of lectures at the University of Chicago in the summer of 1955, where I visited as a special summer student. From the time that I joined the Harvard faculty in 1959, George has given me access to his voluminous handwritten notes on mathematical physics, and has, on occasion, written me long letters explaining various points. Much of his influence can be felt in the first half of this book.

In 1962, I was invited by Prof. Yuval Ne'eman to give a series of lectures on the topology of Lie groups at his seminar, then held at Nahal Soreq. This was after the prediction of the existence of the Ω^- particle (by Gell-Mann and by Ne'eman on the basis of $SU(3)$ symmetry in 1961), but before its momentous discovery at Brookhaven National Laboratory in 1964. This series of lectures developed into a lifelong collaboration. Much of my own work in physics has been in collaboration with Prof. Ne'eman, or an outgrowth of the seminars we have held together over the past 32 years.

Let me now describe the contents. The key idea in Chapter 1 is an action of a group on a set, with the concomitant notions of fixed point sets and stabilizer subgroups. We use these notions to clarify the notion of form and habit in a crystal, and to classify the finite subgroups of $O(3)$. Along the way we show that $Sl(2, \mathrm{C})$ is the double cover of the connected component of the Lorentz group, and hence that $SU(2)$ is a double cover of the three-dimensional rotation group, facts that are central to the understanding of the concept of spin. We conclude with a discussion of icosahedral symmetry in conjunction with the newly discovered carbon molecules, the buckyballs.

Chapter 2 presents the basic facts in the representation theory of finite groups. The central unifying theme is that of character formulas as fixed point formulas, both in this chapter and the next. Of course, we present these formulas in the purely finite context. But they represent the finite prototypes of the more powerful fixed point formulas in modern analysis, such as the Atiyah–Bott theorem. A partial transition to these formulas is given in Appendix G using differential geometric methods and generalized functions. At the end of the chapter we make a first pass at the representation theory of the symmetric groups. I return to this topic in Chapter 5 and in Appendix C.

Chapter 3 discusses induced representations from the point of view of vector bundles. The motivating example is the study of the vibrational spectrum of a molecule, both in classical and quantum mechanics. The main physical idea is the use of Schur's lemma to determine the number of possible vibrational modes and also to derive the quantum mechanical selection rules that determine which transitions are forbidden. In this latter connection, one needs to use tensor products and what are known in the physics literature as 'tensor operators'. I give the Frobenius theory of the representations of a semidirect product, and describe Wigner's use of this method to obtain the irreducible representations of the symmetry group of special relativity. The chapter includes a careful mathematical discussion of the question of the discrete symmetries of space time such as parity and time reversal. The chapter concludes with the Mackey theorems on induced representations and Mackey's approach to exchange forces.

Chapter 4 makes the transition from finite to compact groups. The Peter–Weyl theorem is stated, but its proof is deferred to Appendix E. The irreducible representations of $SU(2)$ and $SO(3)$ are derived, with the concomitant theory of spherical harmonics. Applications include a discussion of the hydrogen spectrum and the role of the representation theory of the rotation group in the periodic table and the magic numbers of nuclear physics. I discuss the role of the Clebsch–Gordan coefficients in isospin, in particular in pion–nucleon scattering experiments, and show how the Klein–Gordan equation, the Dirac equation, Weyl's neutrino equation and Maxwell's equation are related to the appropriate irreducible representations of the Poincaré group in Wigner's list. A brief introduction to Lie algebra methods is included.

Chapter 5 is devoted to the Schur–Weyl duality between representations of the symmetric groups and the general linear groups. The representations of the special unitary groups are derived from this duality in the standard fashion. The results are applied to the study of quarks. A typical application is the derivation of the nucleon magnetic moments from the quark theory. The physics in this chapter represents discoveries up to the early 1970s. I do not include a discussion of electroweak unification, but end, somewhat out of context, with a discussion of the differential geometry of the Higgs mechanism. I have not included any of the material on grand unified or supersymmetric models. My feeling is that no one of these models has won the day, and that the fundamental problems, such as confinement, the mass spectrum, divergences, the source of the Higgs field, etc., must be regarded as open questions. In the meantime, the attention of much of the theoretical physics community has turned elsewhere.

There are seven appendices. Appendix A takes the study of mathematical crystallography a bit further than does the treatment in Chapter 1. It does not go through the detailed classification of the crystallographic groups, but does give a description of how this classification proceeds. In particular, it gives a precise definition of the Bravais lattices and their classification. My feeling is that this material is of general cultural importance (and is, of course, central to solid state physics) but too technical to be included in the introductory first chapter.

Appendix B provides the necessary background on tensor products, as this material is not always included in the standard linear algebra course.

Appendix C provides proofs of some of the more technical aspects of the representation theory of the symmetric group. I chose to follow the beautiful 1977 paper of James. This method illustrates the power of the Gel'fand approach to integral geometry. Alternative approaches to this theory, such as via Hopf algebras or combinatorics, have their own individual merits, and are available elsewhere.

Appendix D gives the proof (following Bargmann) of Wigner's theorem on the symmetries of quantum logic. This theorem lies at the heart of the application of group theory to quantum mechanics. It is the quantum mechanical version of the fundamental theorem of projective geometry. As it is not usually included in the standard texts, I thought it important to include it here.

Appendix E gives the proofs of the basic facts about the representations of compact groups. The treatment is concise and standard, and practically no hard theorems in functional analysis are used. Nevertheless, I felt that it would be too much of a distraction from the main storyline to include this material in the main text.

Appendix F includes no mathematics at all. It is devoted to a history of 19th century spectroscopy. Many quantum mechanics texts start with a little of the prehistory of the subject, usually beginning with the Bohr atom. But it took a century of research to reach Bohr's epoch making paper, and during most of that period the existence of atoms was in dispute, not to say the existence of subatomic constituents. My feeling is that we are in a similar state today with regards to quarks and their possible constituent components. So a look back at how the science of spectroscopy actually progressed might be a source of comfort and amusement during our present period of groping towards an understanding of the deeper components of matter. My guides to the original literature were early Encyclopedia Britannica articles, all written by key players (that is how it was in those days) and the excellent unpublished Ph.D. thesis by Clifford Lawrence Meier entitled 'The role of spectroscopy in the acceptance of an internally structured atom 1860–1920', submitted to the University of Wisconsin in 1964. Of course, I bear the responsibility for the judgment calls in the shaping of the story.

Appendix G is taken from my joint book *Geometric Asymptotics*, written with Victor Guillemin. I try to give a taste of how the fixed point theorems given in the text in the finitistic setting can be formulated and proved in the framework of differential topology.

A word about prerequisites. I have tried to make the demands on the mathematical background of the reader as modest as possible. A course in multivariate calculus and linear algebra, together with an elementary physics course, should suffice. Especially if the reader will forgive my occasional lapses into more advanced material.

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