1

BASIC DEFINITIONS AND EXAMPLES

1.1 Groups: definition and examples

In this chapter we will introduce the basic mathematical concepts associated with symmetry: the notion of a group and the action of a group on a set. A group G is a set on which we are given a binary operation which behaves much like ordinary multiplication; that is, we are given a map of $G \times G \to G$ sending the pair (p, q) into pq, satisfying the associative law, the existence of an identity element e, and the existence of an inverse. That is, we assume that

- $(pq)r = p(qr)$ for any three elements p, q, r in G;
- there exists an element, e, in G such that $ep = pe = p$ for all p in G; and
- for every p in G there is a p^{-1} in G such that $pp^{-1} = p^{-1}p = e$.

Example 1

(a) Let $\mathbb{Z}_4$ denote the additive group of the integers modulo 4. The elements of this group are equivalence classes which we shall call e, a, b and c:

$$\begin{aligned} e &= \{0, \quad 4, \quad -4, \quad 8, \quad -8, \quad \ldots\} \\ a &= \{1, \quad 5, \quad -3, \quad 9, \quad -7, \quad \ldots\} \\ b &= \{2, \quad 6, \quad -2, \quad 10, \quad -6, \quad \ldots\} \\ c &= \{3, \quad 7, \quad -1, \quad 11, \quad -5, \quad \ldots\}. \end{aligned}$$

The binary operation is addition modulo 4; for example, since $1 + 3 = 4$, which equals 0 modulo 4, we have $ac = e$. The identity element is e. Since $ac = e$, $a^{-1} = c$ and $c^{-1} = a$; since $bb = e, b^{-1} = b$.

(b) Let G denote the following set of four 2×2 real matrices:

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad b = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The binary operation is matrix multiplication; for example,

$$ac = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = e.$$

The identity element is the identity matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and the inverse of each element is its matrix inverse; for example,

$$a^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = c.$$

(c) Let C_4 denote the group of rotational symmetries of the square, as follows:

$e =$ identity (rotation through 0)
$a =$ counterclockwise rotation through $\pi/2$
$b =$ counterclockwise rotation through π
$c =$ counterclockwise rotation through $3\pi/2$ (clockwise rotation through $\pi/2$).

Now the group operation is composition of transformations. Clearly the 'multiplication table' is the same as in the preceding two examples; we have considered three different realizations of the same abstract group, the so-called 'cyclic group of four elements'. It is a simple example of a *finite* group.

Example 2

We turn now to an example of a group which has an infinite number of elements. Let $SL(2, \mathbb{C})$ denote the set of 2×2 matrices of determinant 1 with complex entries. Thus, an element A of $SL(2, \mathbb{C})$ is given as

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where a, b, c and d are complex numbers satisfying

$$ad - bc = 1.$$

Multiplication is the ordinary multiplication of matrices. Since the determinant of the product of two matrices is the product of their determinants, we see that if A and B are elements of $SL(2, \mathbb{C})$, then so is their product AB. If A is an element of $SL(2, \mathbb{C})$, so that $\det A = 1$, then A is invertible and $\det A^{-1} = 1$, so that A^{-1} exists and lies in $SL(2, \mathbb{C})$. The identity element of the group is the identity matrix, i.e.

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The associative law holds for matrix multiplication and thus $SL(2, \mathbb{C})$ is indeed a group. Notice that the commutative law does not hold in general for this group.

More generally, we can consider $n \times n$ matrices with either real or complex entries. The collection of real invertible $n \times n$ matrices is denoted by $GL(n, \mathbb{R})$. (Notice that here the condition of invertibility has to be added as a supplemental hypothesis. Not all

2×2 or $n \times n$ matrices are invertible, but those that are invertible form a group.) The group $GL(n, \mathbb{R})$ is called the *real general linear* group in n variables. We can also consider the group $SL(n, \mathbb{R})$ consisting of the $n \times n$ real matrices of determinant 1. It is called the *real special linear* group in n variables. Similarly, we can consider the group $GL(n, \mathbb{C})$ of all invertible complex $n \times n$ matrices or the group $SL(n, \mathbb{C})$ of all $n \times n$ complex matrices of determinant 1.

Example 3

As a third example of a group we can consider the group, $O(3)$, of all orthogonal transformations in Euclidean three-dimensional space. This is the group of all linear transformations of three-dimensional space which preserve the Euclidean distance; that is, those transformations, A, which satisfy

$$\|A\mathbf{v}\| = \|\mathbf{v}\|$$

for all vectors $\mathbf{v}$ in ordinary three-dimensional space. If we choose an orthonormal basis for three-dimensional space so that every A becomes identified with a matrix, then A is an orthogonal transformation if and only if

$$AA^{t} = e,$$

where e denotes the identity matrix in three dimensions. Notice that this equation is the same as $A^{t} = A^{-1}$. We see immediately that the product of any two orthogonal transformations is again orthogonal and that the inverse of any orthogonal transformation exists and is orthogonal. Thus, the collection of all orthogonal transformations does indeed form a group. Since $\det A = \det A^{t}$, it follows from $AA^{t} = e$ that $(\det A)^2 = 1$. Thus, for any orthogonal transformation A we have $\det A = \pm 1$. The collection of those matrices which are orthogonal, and which satisfy the further condition that $\det A = +1$, forms a subcollection of $O(3)$, which in itself is a group and which we will denote by $SO(3)$. We say that $SO(3)$ is a *subgroup* of $O(3)$. $SO(3)$ is called the special orthogonal group in three variables. (Similarly, $SL(n, \mathbb{C})$ is a subgroup of $GL(n, \mathbb{C})$, and $SL(n, \mathbb{R})$ is a subgroup of $GL(n, \mathbb{R})$.) More generally, if we put the standard Euclidean scalar product on the n-dimensional space $\mathbb{R}^n$, we can consider the orthogonal group $O(n)$ of all orthogonal $n \times n$ matrices and the corresponding subgroup $SO(n)$ of those orthogonal matrices with determinant 1.

Example 4

Let $\mathbb{C}^n$ denote the n-dimensional complex vector space of all complex n-tuples with its standard Hermitian scalar product, so that

$$(\mathbf{z}, \mathbf{w}) = z_1 \bar{w}_1 + \cdots + z_n \bar{w}_n,$$

where

$$\mathbf{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}.$$

Table 1.

	1	-1
1	1	-1
-1	-1	1

Table 2.

	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

A complex matrix A is *unitary* if

$$(A\mathbf{z}, A\mathbf{w}) = (\mathbf{z}, \mathbf{w})$$

for all $\mathbf{z}$ and $\mathbf{w}$ in $\mathbb{C}^n$. If we denote $\overline{A^t}$ (the complex conjugate transpose of A) by A^*, we may say that A is unitary only if $AA^* = e$. The product of two unitary matrices is unitary, and the inverse of a unitary matrix is unitary; so the collection of unitary $n \times n$ matrices forms a group which we denote by $U(n)$. Since $\det A^* = \overline{\det A}$, we see that $|\det A| = 1$ for A in $U(n)$. The subgroup of $U(n)$ consisting of those matrices which in addition satisfy $\det A = 1$ is denoted by $SU(n)$.

Thus, for example, the group $SU(2)$ consists of all 2×2 matrices of the form

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad \text{where} \quad |a|^2 + |b|^2 = 1.$$

Example 5

We can generalize Examples 1(*a*), (*b*) and (*c*) by replacing the number 4 by any positive integer. For instance, we can consider the group C_2 consisting of two elements with the 'multiplication table' as in Table 1, which is isomorphic to the additive group of the integers modulo 2. Similarly, we can think of the three-element group, C_3, with elements $1, \omega, \omega^2$ where $\omega = \exp 2\pi i/3$ which obey the 'multiplication table' shown in Table 2.

The group C_3 can be thought of as the additive group of the integers modulo 3, or as the group of all rotations in the plane which preserve an equilateral triangle centered at the origin. Thus, ω represents rotation through $2\pi/3 = 120°$.

We have already considered the group C_4 of all rotations preserving a square. It

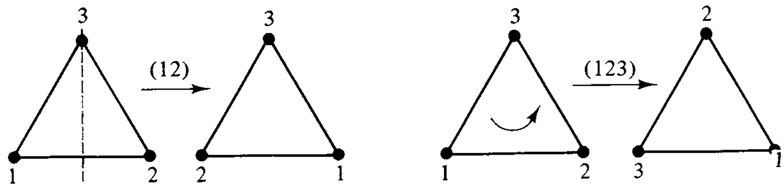

Fig. 1.1

contains four elements, consisting of the identity, rotation through $\pi/2$, rotation through π, and rotation through $3\pi/2$. We can now recognize that C_4 is a subgroup of $SO(2)$, the group of all rotations in the plane. More generally, we can consider C_n as the group of all rotations which preserve a regular polygon with n sides. It will consist of the identity and all rotations through angles of the form $2\pi k/n$.

Example 6

Let us go back to the equilateral triangle. We can consider the group of *all* symmetries of the triangle, not only the rotations. That is, we can allow reflection about perpendicular bisectors as well. This group has six elements; we will denote it by S_3. Notice that we can find some element in S_3 which has the effect of making any desired permutation of the vertices of the triangle. Let us denote the vertices of the triangle by 1, 2 and 3 (Fig. 1.1). Suppose, for example, that (12) denotes the permutation which interchanges the vertices 1 and 2 but leaves a third vertex, 3, fixed. This permutation can be achieved by a reflection about the perpendicular bisector of the edge joining 1 to 2.

Similarly, let (123) denote the permutation that sends 1 into 2, 2 into 3, and 3 into 1. This can be achieved by rotating the triangle through 120°. The permutation (132), which sends 1 into 3, 3 into 2, and 2 into 1, is achieved by rotating the triangle through 240°. From this we see that the group of symmetries of an equilateral triangle is the same as the group of all permutations on three symbols.

Suppose we consider four symbols 1, 2, 3, 4, instead of three. Let s be a one-to-one map of this four-element set onto itself. Thus, s is a permutation of this four-element set. There are four possibilities for $s(1)$: it can be any of the numbers 1, 2, 3, 4. Once we know what $s(1)$ is, then there are three remaining possibilities for $s(2)$, then two remaining possibilities for $s(3)$. Finally, $s(4)$ will be completely determined by being the last remaining number. Thus, there are $4 \cdot 3 \cdot 2 \cdot 1 = 4! = 24$ permutations on four letters. The group S_4 is the group of all these permutations. Similarly, we define the group S_n to be the group of all permutations; that is, all one-to-one transformations on a set with n elements.

Example 7

As a final example, we consider the group of all symmetries of the square, denoted by D_4. D_4 contains eight elements: four rotations, together with four reflections – the reflections about the two diagonals, and the reflections about the two perpendicular bisectors (see Fig. 1.2).

Each element of D_4 permutes the vertices 1, 2, 3, 4 of the square. Thus, we may regard

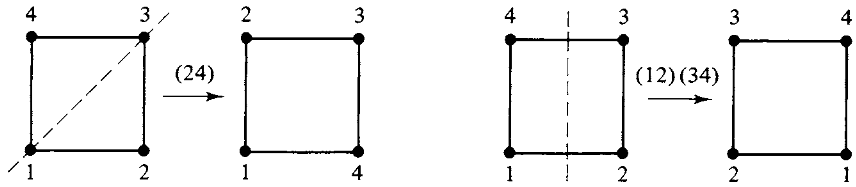

Fig. 1.2

D_4 as a subgroup of S_4, but not every element of S_4 (which has 24 elements all together) lies in D_4, which has only eight elements. Similarly, the group D_n, the group of symmetries of the regular polygon with n sides, is a subgroup of the group S_n of permutations of n symbols. The reader should check that D_n contains $2n$ elements.

1.2 Homomorphisms: the relation between $SL(2, \mathbb{C})$ and the Lorentz group

Let G_1 and G_2 be groups. Let ϕ be a map from G_1 to G_2. We say that ϕ is a *homomorphism* if

$$\phi(ab) = \phi(a)\phi(b) \quad \text{for all } a \text{ and } b \text{ in } G_1.$$

The notion of homomorphism is central to the study of groups and so we give some examples. Take $G_1 = \mathbb{Z}$ to be the integers and $G_2 = C_2$. Define the map ϕ by

$$\phi(n) = (-1)^n.$$

Recall that group 'multiplication' is ordinary addition in $\mathbb{Z}$ so that the condition that ϕ be a homomorphism reduces to the assertion that

$$\phi(a + b) = \phi(a)\phi(b),$$

i.e that

$$(-1)^{a+b} = (-1)^a(-1)^b$$

which is clearly true. More generally, we can define a homomorphism from $\mathbb{Z}$ to C_k by

$$\phi(a) = \exp 2\pi i a/k = \omega^a, \quad \text{where } \omega \text{ equals } \exp 2\pi i/k.$$

This generalizes the construction of Example 1 of the preceding section. Basically, what the homomorphism ϕ is telling us is that we can regard multiplication in C_k as 'addition modulo k' in the integers.

We now want to describe another homomorphism which has many important physical applications and which will recur frequently in the rest of this book. For this we need to introduce still another group, the Lorentz group. Let M denote the four-dimensional space $M = \mathbb{R}^4$, with the 'Lorentz metric'

$$\|\mathbf{x}\|^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2, \quad \text{where } \mathbf{x} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Thus, M is the ordinary Minkowski space of special relativity, where we have chosen units in which the speed of light is unity. A Lorentz transformation, B, is a linear transformation of M into itself which preserves the Lorentz metric, i.e. which satisfies

$$\|B\mathbf{x}\|^2 = \|\mathbf{x}\|^2, \quad \text{for all } \mathbf{x} \text{ in } M.$$

We let L denote the group of all Lorentz transformations; L is called the Lorentz group.

We now describe a homomorphism from the group $SL(2, \mathbb{C})$ to the group L. For this purpose we shall identify every point $\mathbf{x}$ in M with a two-by-two self-adjoint matrix, as follows:

$$x = \begin{pmatrix} x_0 + x_3 & x_1 - \mathrm{i}x_2 \\ x_1 + \mathrm{i}x_2 & x_0 - x_3 \end{pmatrix} \quad \text{represents } \mathbf{x} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Notice that

$$x^* = \overline{x^{\mathrm{t}}} = x,$$

and that

$$\det x = \|\mathbf{x}\|^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2.$$

Indeed, the most general self-adjoint 2×2 matrix can be written in this form: if $x = x^*$ is a self-adjoint matrix, then its diagonal entries must be real. We can let $x_0 = \frac{1}{2}\operatorname{tr} x = \frac{1}{2}$(the sum of the diagonal entries of x) and similarly $x_3 = \frac{1}{2}$(the difference of the diagonal entries of x). Also, we can write the entry in the lower left-hand corner of x as $x_1 + \mathrm{i}x_2$. Then the entry in the upper right-hand corner will be $x_1 - \mathrm{i}x_2$. In effect, what we have done is to note that the collection of 2×2 self-adjoint matrices is a four-dimensional real vector space, for which a convenient basis consists of the identity matrix $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the three so-called 'Pauli matrices'

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We have identified the vector

$$\mathbf{x} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

with the matrix

$$x = x_0 e + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3.$$

Now let A be any 2×2 matrix. We define the action of the matrix A on the self-adjoint matrix x by

$$x \to AxA^*$$

and we denote the corresponding action on the vector $\mathbf{x}$ by $\phi(A)\mathbf{x}$. Notice that $(AxA^*)^* = A^{**}x^*A^* = AxA^*$, so that AxA^* is again self-adjoint. Notice also that

$$\det(AxA^*) = |\det A|^2 \det x.$$

Therefore, if A is in $SL(2, \mathbb{C})$, then

$$\| \phi(A)\mathbf{x} \|^2 = \| \mathbf{x} \|^2,$$

so that, if A is in $SL(2, \mathbb{C})$, $\phi(A)$ represents a Lorentz transformation. Notice also that

$$ABx(AB)^* = ABxB^*A^* = A(BxB^*)A^*$$

so that

$$\phi(AB)\mathbf{x} = \phi(A)\phi(B)\mathbf{x}.$$

Thus, $\phi(AB) = \phi(A)\phi(B)$, so that ϕ is a homomorphism. Notice, however, that $\phi(-A) = \phi(A)$ so that ϕ is not one-to-one. The matrices A and $-A$ correspond to the same Lorentz transformation.

Suppose now that A belongs to the subgroup $SU(2)$ of $SL(2, \mathbb{C})$. This means that A is a unitary matrix, satisfying

$$AA^* = I; \quad \text{i.e.} \quad AIA^* = I.$$

Therefore, if $\mathbf{e}_0$ denotes the vector

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

which is represented by the 2×2 identity matrix I, then

$$\phi(A)\mathbf{e}_0 = \mathbf{e}_0.$$

If a Lorentz transformation C satisfies $C\mathbf{e}_0 = \mathbf{e}_0$, then C also carries the three-dimensional space $e_0^{\perp}$, consisting of vectors

$$\begin{pmatrix} 0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

into itself and C is an orthogonal transformation on that three-dimensional space. Put another way, we can regard $O(3)$ as the subgroup of L consisting precisely of those Lorentz transformations which satisfy $C\mathbf{e}_0 = \mathbf{e}_0$. Thus, the mapping ϕ, when restricted to $SU(2)$, maps $SU(2)$ into $O(3)$.

For example, let us consider the diagonal matrix

$$U_\theta = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}.$$

The $\phi(U_\theta)\mathbf{x}$ may be computed by matrix multiplication as follows:

$$U_\theta x U_{-\theta} = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

$$= \begin{pmatrix} x_0 + x_3 & e^{-2i\theta}(x_1 - ix_2) \\ e^{2i\theta}(x_1 + ix_2) & x_0 - x_3 \end{pmatrix}.$$

Thus, $\phi(U_\theta)$ leaves x_0 and x_3 unchanged and hence is a rotation about the x_3 axis. Since it sends $x_1 + ix_2$ into $e^{2i\theta}(x_1 + ix_2)$, we see that it is a rotation through angle 2θ in the x_1, x_2 plane. We have thus shown that

$\phi(U_\theta)$ is rotation through angle 2θ about the x_3 axis.

Notice that as θ ranges from 0 to π the corresponding rotation goes from 0 to 2π, making a complete circuit. As θ ranges from 0 to 2π, the corresponding rotation goes through *two* complete circuits. This is a reflection of the fact that $\phi(-A) = \phi(A)$.

Similarly, consider the action of the unitary matrix

$$V_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \quad \text{for which} \quad V_\alpha^* = V_{-\alpha}.$$

We calculate $\phi(V_\alpha)\mathbf{x}$ by matrix multiplication as follows:

$$V_\alpha x V_{-\alpha} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}.$$

We can easily determine the action of $\phi(V_\alpha)$ on the vector

$$\mathbf{e}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

by taking $x_0 = x_1 = x_3 = 0$, so that

$$x = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

We find that $\phi(V_\alpha)\mathbf{e}_2 = \mathbf{e}_2$, so that $\phi(V_\alpha)$ must be a rotation about the x_2 axis.

We now determine the action of $\phi(V_\alpha)$ on the basis vector

$$\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

by taking $x = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We find

$$V_\alpha \sigma_3 V_{-\alpha} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$$

which corresponds to the vector

$$\begin{pmatrix} 0 \\ \sin 2\alpha \\ 0 \\ \cos 2\alpha \end{pmatrix}.$$

We conclude that

$$\phi(V_\alpha)\mathbf{e}_3 = \mathbf{e}_3 \cos 2\alpha + \mathbf{e}_1 \sin 2\alpha$$

so that V_α represents rotation through angle 2α about the x_2 axis.

As a third example, consider the diagonal matrix with real entries

$$M_r = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}.$$

Since $M_r = M_r^*$, we have

$$M_r \times M_r^* = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}\begin{pmatrix} x_0 + x_3 & x_1 + \mathrm{i}x_2 \\ x_1 - \mathrm{i}x_2 & x_0 - x_3 \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} r^2(x_0 + x_3) & x_1 + \mathrm{i}x_2 \\ x_1 - \mathrm{i}x_2 & r^{-2}(x_0 - x_3) \end{pmatrix}.$$

Thus the Lorentz transformation $\phi(M_r)$ leaves x_1 and x_2 alone whereas the x_0 and the x_3 coordinates are transformed into

$$x_0' = \tfrac{1}{2}(r^2 + r^{-2})x_0 + \tfrac{1}{2}(r^2 - r^{-2})x_3 \quad \text{and} \quad x_3' = \tfrac{1}{2}(r^2 - r^{-2})x_0 + \tfrac{1}{2}(r^2 + r^{-2})x_3.$$

We recall the definition of the hyperbolic functions:

$$\cosh u = \tfrac{1}{2}(\mathrm{e}^u + \mathrm{e}^{-u}) \quad \text{and} \quad \sinh u = \tfrac{1}{2}(\mathrm{e}^u - \mathrm{e}^{-u}).$$

The *Lorentz boost* in the z direction with parameter t, denoted by L_t, is defined as the transformation given by

$$x_1' = x_1,\ x_2' = x_2,\ x_0' = (\cosh t)\,x_0 + (\sinh t)x_3 \quad \text{and} \quad x_3' = (\sinh t)x_0 + (\cosh t)x_3.$$

In other words, L_t^z is the Lorentz transformation given by the matrix

$$L_t^z = \begin{pmatrix} \cosh t & 0 & 0 & \sinh t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh t & 0 & 0 & \cosh t \end{pmatrix}.$$

If we set $r = \mathrm{e}^t$ then our preceding computation shows that

$$\phi(M_{\mathrm{e}^t}) = L_{2t}^z.$$

To summarize: let R_θ^z denote rotation through angle θ about the z axis, let R_θ^y denote rotation through angle θ about the y axis. We have shown that

$$\phi(U_\theta) = R_{2\theta}^z,\ \phi(V_\theta) = R_{2\theta}^y \quad \text{and} \quad \phi(M_{\mathrm{e}^t}) = L_{2t}^z.$$