Introduction

Quantum groups first arose in the physics literature, particularly in the work of L. D. Faddeev and the Leningrad school, from the 'inverse scattering method', which had been developed to construct and solve 'integrable' quantum systems. They have excited great interest in the past few years because of their unexpected connections with such, at first sight, unrelated parts of mathematics as the construction of knot invariants and the representation theory of algebraic groups in characteristic p.

In their original form, quantum groups are associative algebras whose defining relations are expressed in terms of a matrix of constants (depending on the integrable system under consideration) called a quantum R-matrix. It was realized independently by V. G. Drinfel'd and M. Jimbo around 1985 that these algebras are Hopf algebras, which, in many cases, are deformations of 'universal enveloping algebras' of Lie algebras. A little later, Yu. I. Manin and S. L. Woronowicz independently constructed non-commutative deformations of the algebra of functions on the groups $SL_2(\mathbb{C})$ and SU_2, respectively, and showed that many of the classical results about algebraic and topological groups admit analogues in the non-commutative case.

Thus, although many of the fundamental papers on quantum groups are written in the language of integrable systems, their properties are accessible by more conventional mathematical techniques, such as the theory of topological and algebraic groups and Lie algebras. Our aim in this book is to present the theory of quantum groups from this latter point of view. In fact, we shall concentrate on the study of the 'Lie algebras' of quantum groups, which seems to be the approach which has proved most powerful, particularly in applications, but we shall also discuss, in rather less detail, their relation with 'non-commutative algebraic geometry and topology'.

We shall now describe what a quantum group is, beginning by trying to explain the motivation for the use of the adjective 'quantum'.

In classical mechanics, the phase space M of a dynamical system is a *Poisson manifold*. This means that the space $\mathcal{F}(M)$ of (differentiable) complex-valued functions on M is equipped with a Lie bracket $\{\ ,\ \} : \mathcal{F}(M) \times \mathcal{F}(M) \to \mathcal{F}(M)$ (satisfying certain additional conditions), called the Poisson bracket. The dynamical equations defining the time evolution of the system are equivalent to the equations

$$\frac{d}{dt} f(m(t)) = \{\mathcal{H}_{\rm cl}, f\}(m(t))$$

for $f \in \mathcal{F}(M)$, where $\mathcal{H}_{\rm cl}$ is a fixed function on M called the (classical) hamiltonian, and $m(t) \in M$ is the 'state' of the system at time t. For example, for a single particle moving along the real line, M is the cotangent bundle $T^*(\mathbb{R})$, and if q is the coordinate on $\mathbb{R}$ ('position') and p the coordinate in the fibre direction ('momentum'), the Poisson bracket is

$$\{f_1, f_2\} = \frac{\partial f_1}{\partial p}\frac{\partial f_2}{\partial q} - \frac{\partial f_2}{\partial p}\frac{\partial f_1}{\partial q}.$$

In particular, the Poisson bracket of the coordinate functions is

(1) $$\{p, q\} = 1.$$

In quantum mechanics, the space M is replaced by the set of rays in a complex Hilbert space V, and the space $\mathcal{F}(M)$ of functions on M by the algebra $\mathrm{Op}(V)$ of (not necessarily bounded) operators on V. The time evolution of an operator A is given by

$$\frac{dA}{dt} = [\mathcal{H}_{\mathrm{qu}}, A]$$

for some operator $\mathcal{H}_{\mathrm{qu}} \in \mathrm{Op}(V)$, called the (quantum) hamiltonian. For example, in the case of a single particle moving along the real line, V is the space $\mathrm{L}^2(\mathbb{R})$ of square-integrable functions of q, and the operators P and Q corresponding to the coordinate functions p and q are given by

$$P = -\sqrt{-1}h\frac{\partial}{\partial q}, \qquad Q = \text{multiplication by } q,$$

where h is $1/2\pi$ times Planck's constant. Note that

(2) $$[P, Q] = -\sqrt{-1}h\,\mathrm{id}_V.$$

The question is: how to pass from the classical to the quantum description of a system. This is the problem of *quantization*. Ideally, one would like a map $\mathcal{Q}$ which assigns to each function $f \in \mathcal{F}(M)$ an operator $\mathcal{Q}(f)$ on V. Moreover, since time evolution in the classical and quantum descriptions is given by taking the Poisson bracket and commutator with the hamiltonian, respectively, $\mathcal{Q}$ should satisfy the relation

$$\mathcal{Q}\{f_1, f_2\} = \frac{[\mathcal{Q}(f_1), \mathcal{Q}(f_2)]}{-\sqrt{-1}h}$$

(the normalization comes from (1) and (2)). Unfortunately, it is known that, even for the simplest case of a single particle moving along the real line, no such map $\mathcal{Q}$ exists.

There is, however, an alternative formulation of the quantization problem, introduced by J. E. Moyal in 1949. This begins by noting that the fundamental difference between the classical and quantum descriptions is that

$\mathcal{F}(M)$ is a commutative algebra, whereas $\mathrm{Op}(V)$ is non-commutative (when $\dim(V) > 1$). Moyal's idea is to try to reproduce the results of quantum mechanics by replacing the usual product on $\mathcal{F}(M)$ by a non-commutative product $*_h$, depending on a parameter h, such that $*_h$ becomes the usual product as $h \to 0$, just as 'quantum mechanics becomes classical mechanics as Planck's constant tends to zero', and such that

$$\lim_{h\to 0} \frac{f_1 *_h f_2 - f_2 *_h f_1}{h} = \{f_1\,, f_2\}. \tag{3}$$

If we think of $\mathcal{F}(M)$ with the Moyal product $*_h$ as a non-commutative algebra of functions $\mathcal{F}_h(M)$, we find ourselves in the realm of non-commutative geometry in the sense of A. Connes. The philosophy here is that any 'space' is determined by the algebra of functions on it (with the usual product). For example, every affine algebraic variety over $\mathbb{C}$ is determined (up to isomorphism) by the commutative algebra of regular functions on it, whereas every compact topological space is determined by its commutative C*-algebra of complex-valued continuous functions. More precisely, the category of 'spaces' in these examples is dual to the category of the corresponding algebras. Thus, a non-commutative algebra should be viewed as the space of functions on a 'non-commutative space', and we can say that Moyal's construction gives a deformation of the classical phase space M to a family of non-commutative (or 'quantum') spaces M_h such that $\mathcal{F}_h(M)$ is the algebra of functions on M_h.

The category of quantum spaces, then, might be defined as the category dual to the category of associative, but not necessarily commutative, algebras. To define the notion of a quantum group, let us first return for a moment to the classical situation. If G is a group, the multiplication $\mu : G \times G \to G$ of G induces a homomorphism $\mu^* = \Delta : \mathcal{F}(G) \to \mathcal{F}(G \times G)$ of algebras of functions. Now, if we define the algebra $\mathcal{F}(G)$ and the tensor product appropriately, $\mathcal{F}(G \times G)$ will be isomorphic to $\mathcal{F}(G) \otimes \mathcal{F}(G)$ as an algebra. For example, if G is an affine algebraic group over $\mathbb{C}$, and $\mathcal{F}(G)$ is the algebra of regular functions on G, the ordinary algebraic tensor product will do. Thus, we have a comultiplication $\Delta : \mathcal{F}(G) \to \mathcal{F}(G) \otimes \mathcal{F}(G)$. (The reason for this terminology is that the multiplication on $\mathcal{F}(G)$ can be viewed as a map $\mathcal{F}(G) \otimes \mathcal{F}(G) \to \mathcal{F}(G)$.) Similarly, the inverse map $\iota : G \to G$ induces a map $\iota^* = S : \mathcal{F}(G) \to \mathcal{F}(G)$, called the antipode, and evaluation at the identity element of G is a homomorphism $\epsilon : \mathcal{F}(G) \to \mathbb{C}$, called the counit. The maps Δ, S and ϵ satisfy certain compatibility properties which reflect the defining properties of the inverse and the associativity of multiplication in G, and combine to give $\mathcal{F}(G)$ the structure of a *Hopf algebra*.

We might therefore *define the category of quantum groups to be the category dual to the category of (not necessarily commutative) Hopf algebras.* (We said 'might' here, and in our tentative definition of a quantum space, because, to ensure that the categories of quantum spaces and quantum groups have reasonable properties, it would be necessary to impose some restrictions on the class of algebras which are acceptable as 'quantized algebras of functions'. Manin suggests that one should work with 'Koszul algebras', but we shall not discuss this point here.) As is common practice in the literature, we shall often abuse terminology by referring to a Hopf algebra itself as a quantum group.

As the preceding discussion suggests, one way to try to construct non-classical examples of quantum groups is to look for deformations, in the category of Hopf algebras, of classical algebras of functions $\mathcal{F}(G)$. Just as the classical Poisson bracket can be recovered as the 'first order part' of Moyal's deformation (see (3)), so it turns out that the existence of a deformation $\mathcal{F}_h(G)$ of $\mathcal{F}(G)$ automatically endows the group G itself with extra structure, namely that of a *Poisson–Lie group*. This is a Poisson structure on G which is compatible with the group structure in a certain sense. Conversely, to construct deformations of $\mathcal{F}(G)$, it is natural to begin by describing the possible Poisson–Lie group structures on G and then to attempt to extend these 'first order deformations' to full deformations. This is the approach taken in this book. Poisson–Lie groups are also of interest in their own right, for they form the natural setting for the study of classical integrable systems with symmetry.

There is another Hopf algebra associated to any Lie group G, namely the universal enveloping algebra $U(\mathfrak{g})$ of its Lie algebra $\mathfrak{g}$. This is essentially the *dual* of $\mathcal{F}(G)$ in the category of Hopf algebras. In general, the vector space dual A^* of any finite-dimensional Hopf algebra A is also a Hopf algebra: the multiplication $A^* \otimes A^* \to A^*$ is dual to the comultiplication $\Delta : A \to A \otimes A$ of A, and the comultiplication of A^* is dual to the multiplication of A. Note that A^* is commutative if and only if A is cocommutative, i.e. if and only if $\Delta(A)$ is contained in the symmetric part of $A \otimes A$. If, as is usually the case in examples of interest, A is infinite dimensional, this duality often continues to hold provided the dual and tensor product are defined appropriately. To a deformation $\mathcal{F}_h(G)$ of $\mathcal{F}(G)$ through (not necessarily commutative) Hopf algebras therefore corresponds a deformation $U_h(\mathfrak{g})$ of $U(\mathfrak{g})$ through (not necessarily cocommutative) Hopf algebras.

In fact, only non-cocommutative deformations of $U(\mathfrak{g})$ are of interest, since any deformation of $U(\mathfrak{g})$ through cocommutative Hopf algebras is necessarily of the form $U(\mathfrak{g}_h)$ for some deformation $\mathfrak{g}_h$ of $\mathfrak{g}$ through Lie algebras. However, many interesting Lie algebras have no non-trivial deformations. This is the case, for example, if $\mathfrak{g}$ is a (finite-dimensional) complex semisimple Lie algebra, such as the Lie algebra $sl_2(\mathbb{C})$ of 2×2 complex matrices of trace zero. This follows from the fact that the condition of semisimplicity is open, so that any small deformation of $\mathfrak{g}$ will still be semisimple, whereas the semisimple Lie algebras are discretely parametrized (by their Dynkin diagrams, for example).

The first example of a non-cocommutative deformation of this type was discovered by P. P. Kulish and E. K. Sklyanin in 1981 in the case $\mathfrak{g} = sl_2(\mathbb{C})$ (although the importance of its Hopf structure was not realized until later). Note that $sl_2(\mathbb{C})$ has a basis

$$\bar{X}^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \bar{X}^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \bar{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4}$$

whose Lie brackets are given by

$$[\bar{X}^+, \bar{X}^-] = \bar{H}, \quad [\bar{H}, \bar{X}^\pm] = \pm 2\bar{X}^\pm. \tag{5a}$$

The comultiplication is given on these basis elements by

$$\Delta(\bar{H}) = \bar{H} \otimes 1 + 1 \otimes \bar{H}, \quad \Delta(\bar{X}^\pm) = \bar{X}^\pm \otimes 1 + 1 \otimes \bar{X}^\pm, \tag{5b}$$

an assignment which extends uniquely to an algebra homomorphism $\Delta : U(sl_2(\mathbb{C})) \to U(sl_2(\mathbb{C})) \otimes U(sl_2(\mathbb{C}))$. The deformation $U_h(sl_2(\mathbb{C}))$ is generated by elements H, $X^\pm$, which satisfy the relations

$$X^+X^- - X^-X^+ = \frac{e^{hH} - e^{-hH}}{e^h - e^{-h}}, \quad HX^\pm - X^\pm H = \pm 2X^\pm. \tag{6a}$$

It has a non-cocommutative comultiplication given on generators by

$$\begin{gathered} \Delta(H) = H \otimes 1 + 1 \otimes H, \\ \Delta(X^+) = X^+ \otimes e^{hH} + 1 \otimes X^+, \quad \Delta(X^-) = X^- \otimes 1 + e^{-hH} \otimes X^-. \end{gathered} \tag{6b}$$

Formally, at least, it is clear that (6a) and (6b) go over into (5a) and (5b) as $h \to 0$. The Hopf algebra defined in (6a,b) is called 'quantum $sl_2(\mathbb{C})$'. (See Chapter 6 for the formulas for the antipode and counit of $U_h(sl_2(\mathbb{C}))$, and for a way to make sense of expressions such as e^{hH}.)

The Hopf algebra dual to $U_h(sl_2(\mathbb{C}))$, the 'algebra $\mathcal{F}_h(SL_2(\mathbb{C}))$ of functions on quantum $SL_2(\mathbb{C})$', was discovered by L. D. Faddeev and L. A. Takhtajan in 1985. It is the associative algebra generated by elements a, b, c, d with the following multiplicative relations:

$$ab = e^{-h}ba, \quad ac = e^{-h}ca, \quad bd = e^{-h}db, \quad cd = e^{-h}dc, \tag{7}$$

$$bc = cb, \quad ad - da + (e^h - e^{-h})bc = 0, \tag{8}$$

$$ad - e^{-h}bc = 1, \tag{9}$$

and comultiplication

$$\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes d,$$

$$\Delta(c) = c \otimes a + d \otimes c, \quad \Delta(d) = c \otimes b + d \otimes d.$$

Note that, when $h \to 0$, the relations (7), (8) and (9) just say that the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has commuting entries and determinant one. Thus, $\mathcal{F}_h(SL_2(\mathbb{C}))$ is a deformation of the algebra of functions on the group $SL_2(\mathbb{C})$ of 2×2 complex matrices of determinant one.

As we mentioned at the beginning of this introduction, the algebra structure of $\mathcal{F}_h(G)$ can be described by a matrix of constants, namely

$$(10) \qquad R = e^{-h/2} \begin{pmatrix} e^h & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & e^h - e^{-h} & 1 & 0 \\ 0 & 0 & 0 & e^h \end{pmatrix}.$$

In fact, if

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the relations (7) and (8) are equivalent to

$$(11) \qquad (T \otimes 1)(1 \otimes T)R = R(1 \otimes T)(T \otimes 1).$$

Note that $T \otimes 1$ and $1 \otimes T$ do not commute, since the entries of T do not commute (if $h \neq 0$); note also that R is most naturally viewed as an element of $\mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$. It is in the form (11) that quantum groups usually appear in the theory of integrable systems.

For the dual Hopf algebra $U_h(sl_2(\mathbb{C}))$, the quantum R-matrix expresses, as one would expect, the non-cocommutativity of the comultiplication. Namely, let $\Delta^{\mathrm{op}}(x)$ be the result of interchanging the order of the factors in $\Delta(x)$, for any $x \in U_h(sl_2(\mathbb{C}))$. It turns out that there is an invertible element $\mathcal{R} \in U_h(sl_2(\mathbb{C})) \otimes U_h(sl_2(\mathbb{C}))$, called the 'universal R-matrix', such that

$$\Delta^{\mathrm{op}}(x) = \mathcal{R}\Delta(x)\mathcal{R}^{-1}$$

for all $x \in U_h(sl_2(\mathbb{C}))$ (actually, $\mathcal{R}$ is a formal infinite sum of elements of the algebraic tensor product). The relation between $\mathcal{R}$ and R is very simple: the reader will easily verify that, if we replace $\tilde{X}^{\pm}$ and $\tilde{H}$ by $X^{\pm}$ and H in (4), we obtain a matrix representation of $U_h(sl_2(\mathbb{C}))$; applying this representation to $\mathcal{R}$ gives the matrix R.

Quantum groups might have remained a curiosity to the mathematical community at large but for their surprising connections with other parts of mathematics, most notably the theory of invariants of links and 3-manifolds, and the representation theory of Lie algebras in characteristic p.

The former depends on the classical relation between braids and links. Recall that a *braid on m strands* is a collection of m non-intersecting strings in $\mathbb{R}^3$ joining m fixed points in a plane to m fixed points in another parallel plane. Joining corresponding points in the two planes in a standard way associates to any braid a link (called its 'closure'), i.e. a collection of non-intersecting circles in $\mathbb{R}^3$. Joining braids end to end makes the set of isotopy classes of braids into a group $\mathcal{B}_m$. The relation with quantum groups arises because there is a simple way to associate to any quantum R-matrix $R \in \mathrm{End}(V \otimes V)$ a representation ρ_m of $\mathcal{B}_m$ on $V^{\otimes m}$ for all $m \geq 2$. This depends on the fact that R satisfies the *quantum Yang–Baxter equation*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}\,;$$

here, R_{12} means $R \otimes \mathrm{id} \in \mathrm{End}(V^{\otimes 3})$, etc. To obtain an invariant of links, one needs a family of 'traces' $tr_m : \mathrm{End}(V^{\otimes m}) \to \mathbb{C}$ such that $tr_m(\rho_m(b)) = tr_n(\rho_n(b'))$ whenever the closures of the braids $b \in \mathcal{B}_m$ and $b' \in \mathcal{B}_n$ are equivalent links. Thanks to a classical theorem of A. Markov, it is known precisely which pairs (b, b') have the latter property (and for this reason, the tr_m are usually called 'Markov traces'). Using the quantum R-matrix (10) and a suitable Markov trace, one obtains in this way the celebrated *Jones polynomial.* In fact, this is essentially Jones's original construction, except that he obtained his R-matrix by using a 'Hecke algebra' instead of a quantum group (but we shall see that Hecke algebras should probably be regarded as 'quantum' objects).

The application to 3-manifolds is based on the well-known fact that every compact, oriented, connected 3-manifold without boundary can be obtained, up to homeomorphism, by performing surgery on a link in the 3-dimensional sphere. One shows that a cleverly chosen combination of the quantum invariants of this link depends only on the 3-manifold, and not on the choice of the link along which surgery is performed.

The application of quantum groups to representations of Lie algebras in characteristic p is no less remarkable. It makes use of a certain 'standard' deformation $U_h(\mathfrak{g})$ of $U(\mathfrak{g})$, where $\mathfrak{g}$ is any finite-dimensional complex semisimple Lie algebra (and which reduces, when $\mathfrak{g} = sl_2(\mathbb{C})$, to the algebra found by Kulish and Sklyanin). To describe the relation with characteristic p, it is convenient to replace the deformation parameter h by $\epsilon = e^h$, and to write $U_\epsilon(\mathfrak{g})$ for $U_h(\mathfrak{g})$. It then turns out that the representation theory of $U_\epsilon(\mathfrak{g})$ depends crucially on whether ϵ is a root of unity or not. In the latter case, the theory is essentially the same as the representation theory of $\mathfrak{g}$ itself (over $\mathbb{C}$), but in the former it resembles the modular representation theory of $\mathfrak{g}$. This is more than an analogy: if ϵ is a primitive pth root of unity, where p is a prime, there is a ring homomorphism from $U_\epsilon(\mathfrak{g})$ to the enveloping algebra $U_{\mathbb{F}_p}(\mathfrak{g})$

of $\mathfrak{g}$ over the field $\mathbb{F}_p$ of p elements (this is obtained essentially by replacing ϵ by $1 \in \mathbb{F}_p$), and, under certain additional conditions, representations of $U_\epsilon(\mathfrak{g})$ can be 'specialized' to give representations of $U_{\mathbb{F}_p}(\mathfrak{g})$. Thus, both the modular representation theory of $\mathfrak{g}$ and the characteristic zero theory are reflected in the representation theory of the family of Hopf algebras $U_\epsilon(\mathfrak{g})$ (over $\mathbb{C}$). Using this relation, substantial progress has been made toward the solution of several long-standing conjectures in the modular theory.

We should also mention the roles played by quantum groups in physics, which go well beyond their origins in inverse scattering theory. Perhaps the most interesting of these is the relation between quantum groups and conformal and quantum field theories. The first evidence of this was the experimental observation that the 'fusion rules' of certain conformal field theories can be reproduced by considering the decomposition of tensor products of representations of the quantum groups $U_\epsilon(\mathfrak{g})$ when ϵ is a root of unity. Further evidence came from a remarkable theorem of T. Kohno and Drinfel'd on the relation between the Knizhnik–Zamolodchikov (KZ) equation and $U_h(\mathfrak{g})$. The KZ equation is a system of first order partial differential equations for a function of m complex variables with values in the tensor product $V^{\otimes m}$, where V is a representation of $\mathfrak{g}$. The KZ system has regular singularities along the hyperplanes $z_i = z_j$ in $\mathbb{C}^m$, for all $i \neq j$, and can be viewed as a connection on a bundle over the complement $\mathcal{D}_m$ of these hyperplanes, with fibre $V^{\otimes m}$. Moreover, this connection has a symmetry property which means that there is an induced connection on a bundle over the space $\mathcal{C}_m$ of orbits of the obvious action of the symmetric group Σ_m on $\mathcal{D}_m$. The crucial property of the KZ equation is that this connection is flat, which implies that the monodromy of its solutions defines a representation of the fundamental group of $\mathcal{C}_m$ on $V^{\otimes m}$. The latter group is exactly the braid group $\mathcal{B}_m$ that we discussed above, where we noted that a representation of $\mathcal{B}_m$ on $V^{\otimes m}$ could also be obtained by using a quantum R-matrix. According to Kohno and Drinfel'd, if we use the R-matrix given by the action of the universal R-matrix of $U_h(\mathfrak{g})$ on $V \otimes V$, where h is 'generic', these two representations of $\mathcal{B}_m$ coincide.

The importance of the KZ equation in conformal field theory is that it is satisfied by the 'm-point functions' of the theory. Thus, the Kohno–Drinfel'd theorem indicates a connection between $U_h(\mathfrak{g})$ and conformal field theory. This has recently been extended by D. Kazhdan and G. Lusztig, who have shown that the KZ equation is intimately related to the category of representations of the 'specialized' algebras $U_\epsilon(\mathfrak{g})$ when ϵ is a root of unity. This result provides an 'explanation' for the coincidence between the fusion rules arising in conformal field theory and those arising from quantum groups at roots of unity.

These and other examples to be discussed in this book show that the theory of quantum groups occupies an important place in the mainstream of mathematics and mathematical physics.

We now describe the contents of the book systematically. In Chapter 1, we give the definition and basic properties of Poisson–Lie groups, and of their infinitesimal counterparts, Lie bialgebras. We describe, among other examples, a standard family of Lie bialgebra structures on every (finite-dimensional) complex simple Lie algebra. These induce Poisson–Lie group structures on the associated compact Lie group K. Like all Poisson manifolds, K has a canonical decomposition into symplectic submanifolds, called its *symplectic leaves*, and it turns out that they have a beautiful description in terms of the so-called Bruhat decomposition of K.

In the last section of this chapter, we describe the formulation of quantization as a deformation of Poisson structure. We discuss only the simplest example of a single particle moving along the real line, since our aim is mainly to motivate the treatment of the deformation theory of Hopf algebras in Chapter 6.

Chapter 2 returns to the discussion of Lie bialgebras. We show that Lie bialgebra structures on a Lie algebra $\mathfrak{g}$ can be constructed from solutions of the 'classical Yang–Baxter equation' (CYBE), and that in some cases, for example when $\mathfrak{g}$ is complex semisimple, all Lie bialgebra structures arise in this way. We also discuss the relation between the CYBE and classical integrable systems.

Chapter 2 shows the importance of the problem of classifying the solutions of the CYBE. In Chapter 3, we give an essentially complete description of the solutions with values in $\mathfrak{g} \otimes \mathfrak{g}$, where $\mathfrak{g}$ is a complex simple Lie algebra. The discussion requires more familiarity with Lie theory than the other early chapters of the book, and, since most of the results will not be needed later, we recommend that this chapter be omitted on a first reading.

Chapter 4 begins with a summary of the general results about Hopf algebras we shall need. We have already mentioned that quantum groups are usually non-cocommutative as Hopf algebras. However, they (or their duals) are often 'quasitriangular': this means that the comultiplication of the Hopf algebra A is conjugate to the opposite comultiplication by an invertible element $\mathcal{R}$ of $A \otimes A$, called its universal R-matrix. The element $\mathcal{R}$ satisfies certain additional conditions, including the quantum Yang–Baxter equation (QYBE). We discuss the general properties of quasitriangular Hopf algebras and their relation with the QYBE.

One of the most important general properties of a Hopf algebra A is that there is a natural way to make the tensor product of two representations of A into another representation of A. This means that the category $\mathbf{rep}_A$ of representations of A is a *monoidal category*. If A has additional properties, these will be reflected in $\mathbf{rep}_A$ and vice versa. For example, if A is quasitriangular, $\mathbf{rep}_A$ is a quasitensor category, which means roughly that the tensor product operation is associative and commutative up to isomorphism. We discuss the basic properties of quasitensor categories in Chapter 5, and give a number of examples arising from algebra, topology and physics.

The idea that the properties of a Hopf algebra are encoded in, and might be recovered from, its representations is called *Tannaka–Krein duality*, and this will be a guiding principle throughout this book. We prove a Tannaka–Krein type theorem valid for a large class of Hopf algebras.

We meet our first examples of quantum groups in Chapters 6 and 7. We begin by discussing the general theory of deformations of Hopf algebras, concentrating on universal enveloping algebras and function algebras. We show that any deformation $U_h(\mathfrak{g})$ of $U(\mathfrak{g})$ gives rise to a Lie bialgebra structure on $\mathfrak{g}$, and that every finite-dimensional Lie bialgebra arises as the 'classical limit' of some deformation in this way. To make sense of formulas such as those in (6a,b), one interprets $U_h(\mathfrak{g})$ as an algebra over the ring $\mathbb{C}[[h]]$ of formal power series in an indeterminate h.

In Chapter 6, we 'derive' the deformation $U_h(sl_2(\mathbb{C}))$ described above and show that it can be extended to a deformation of the universal enveloping algebra of any symmetrizable Kac–Moody algebra, using the fact that such algebras are generated by certain $sl_2(\mathbb{C})$ subalgebras.

In Chapter 7, we construct deformations of the algebras of functions on the classical complex Lie groups using the quantum R-matrix method outlined above. We also discuss the duality relation between these quantized function algebras and the quantized universal enveloping algebras of Chapter 6. For a large class of quantum R-matrices, we construct an analogue of the de Rham complex, which can be viewed as a theory of 'differential calculus' on the associated quantum group. In the final section of this chapter, we discuss the relation between the quantum Yang–Baxter equation and certain models in statistical mechanics, where the QYBE plays the role of an integrability condition.

In Chapter 8, we obtain a formula for the universal R-matrix of the quantum groups $U_h(\mathfrak{g})$ defined in Chapter 6, assuming that $\mathfrak{g}$ is finite dimensional. We make use of an action of a (generalized) braid group on any quantum group of this type, which is analogous to the well-known action of (a finite covering of) the Weyl group of $\mathfrak{g}$ on $U(\mathfrak{g})$ in the classical situation.

Chapters 9, 10 and 11 are devoted to the structure and representation theory of the 'specialization' $U_\epsilon(\mathfrak{g})$, where $\epsilon \in \mathbb{C}^\times$. The definition of $U_\epsilon(\mathfrak{g})$ requires some effort, for $U_h(\mathfrak{g})$ is defined over the algebra of formal power series in h, so that the only specialization which appears to make sense is $h = 0$. In fact, as we show in Chapter 9, the specialization to $U_\epsilon(\mathfrak{g})$ can be carried out in two ways, called 'restricted' and 'non-restricted'. The two specializations coincide if ϵ is not a root of unity, but not otherwise. We also study the relation, hinted at above, between $U_\epsilon(\mathfrak{g})$ when ϵ is a root of unity and Lie algebras in characteristic p.

In Chapter 10, we give the classification of the finite-dimensional irreducible representations of $U_\epsilon(\mathfrak{g})$ when $\mathfrak{g}$ is a finite-dimensional complex simple Lie algebra and ϵ is not a root of unity. It turns out that, up to a rather trivial twisting by certain outer automorphisms, the parametrization, and even the