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978-0-521-55846-4 - Partial Differential Equations in Classical Mathematical Physics

Isaak Rubinstein and Lev Rubinstein

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The unique characteristic of this book is that it considers the theory of partial differential equations in mathematical physics as the language of continuous processes, that is to say, as an interdisciplinary science that treats the hierarchy of mathematical phenomena as reflections of their physical counterparts. Special attention is drawn to tracing the development of these mathematical phenomena in different natural sciences, with examples drawn from continuum mechanics, electrodynamics, transport phenomena, thermodynamics, and chemical kinetics. At the same time, the authors trace the interrelation between the different types of problems – elliptic, parabolic, and hyperbolic – as the mathematical counterparts of stationary and evolutionary processes. This interrelation is traced through study of the asymptotics of the solutions of the respective initial boundary-value problems both with respect to time and the governing parameters of the problem.

This combination of mathematical comprehensiveness and natural scientific motivation represents a step forward in the presentation of the classical theory of PDEs, one that will be appreciated by both graduate students and researchers alike.

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Preface

This book represents an attempt to implement a general approach that in essence views the theory of partial differential equations (PDEs) of mathematical physics as the language of continuous processes, that is, an interdisciplinary science that considers the hierarchy of mathematical phenomena as a reflection of their physical counterparts. A comprehensive, mathematically rigorous account of the classical theory of PDEs in mathematical physics is thus inseparably bound with the features of the corresponding natural continuum objects. We shall therefore endeavor to trace the simultaneous origins of some basic mathematical objects in different natural contexts (continuum mechanics, electrodynamics, transport phenomena, thermodynamics, and chemical kinetics). In parallel, we shall trace the interrelation between different types of problems (elliptic, parabolic, and hyperbolic) as mathematical counterparts of their natural prototypes: steady-state and evolutionary processes (dissipative and conservative). This will be done by an asymptotic analysis of the behavior of these processes in time and their dependence on the relevant governing parameters.

In view of the almost complete absence of a physics background in undergraduate and graduate curricula of mathematics and applied mathematics, it seems important, in a course of mathematical physics, to provide an introduction to the basic concepts of different natural sciences and their relation with PDEs in terms of certain typical boundary-value problems that recur in different scientific contexts. Chapters 1 and 2 are therefore addressed primarily to students of mathematics. On the other hand, a rigorous and systematic exposition of classical methods of mathematical physics is undoubtedly necessary for future workers in the applied and engineering sciences, particularly in view of the growing sophistication of industry and the increasing use of mathematical methods in natural sciences. In addition, since modern mathematical education virtually ignores certain efficient classical methods of mathematical physics (such as

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potential theory and the use of Fredholm and Volterra integral equations in solving boundary-value problems), some attention to these topics is highly desirable.

To summarize, this book is addressed to graduate students of applied mathematics and of the natural and engineering sciences. Selected parts may form the basis of an undergraduate course in applied mathematics for mathematics, physics, or engineering students. Most of the material in this book was featured in courses given by one of us (L.R.) for graduate students in mathematics and applied mathematics and by the other (I.R.) for undergraduate students of electrical engineering and physics.

Besides the bulk of the text, which is addressed primarily to graduate students, some chapters are addressed to specialists and are rather monographic in nature (i.e., Chapters 16–18).