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Introduction

1. Mathematical physics

Mathematical physics is an interdisciplinary science which, on the basis of the fundamental (mostly phenomenological) laws of physics, uses mathematical methods to study processes evolving in material media. Its purpose is to formulate equations describing a process to within a reasonable degree of idealization (i.e., disregarding details that are not essential for its qualitative and quantitative essences), to develop methods for solution of the resulting problem, and to analyze the qualitative and quantitative properties of the solutions. In this latter respect mathematical physics borders on numerical analysis and mathematical simulation, but in its most important aspects it borders on the theoretical and even experimental natural sciences.

In this book we shall restrict our attention to phenomena of the "macro" world – more precisely, to processes evolving in continuous media. At this point some elaboration of the very notion of a continuous medium is desirable, since at first sight it might seem incompatible with an atomistic view of the universe. The notion of continuous medium is related to the following notion of a physical element of volume. Consider some process evolving in a region $D \subset \mathbb{R}_3$ and let $K \subset D$ be a subset of positive 3-dimensional measure with diameter d:

$$d = \max_{p\in K} \max_{q\in K} r_{pq} \tag{1.1}$$

where r_{pq} is the distance between points p and q. Fixing $p \in K$, consider it as a representative of K. Assume that d is much smaller than the characteristic size d_0 of D (e.g., the upper limit of the diameters of all spheres contained in D), but that the number of individual material particles in K is very large and their maximum size is very small compared with d. Let us consider some physical characteristic F of particles in K (e.g., the velocity $\mathbf{v}$ at time t of the individual particles in K). Let $\hat{F}(p,t)$ denote the value of F averaged over all particles in K. The medium in D may be called *continuous with* *respect to* F if $\hat{F}(p,t)$ is a continuous function of p and t everywhere in D, except for finitely many sets of points of zero 3-dimensional measure (i.e., except for finitely many surfaces, lines, or separate points). If the medium is continuous with respect to all physical parameters of relevance for the process in question, one can speak of the medium as simply continuous.

Processes in nature may be divided, roughly speaking, into three groups: (1) stationary processes, in which the state of the system is independent of time; (2) dissipative time-dependent evolution processes; and (3) conservative evolution processes. Although the processes in group (3) may be viewed as the processes of group (2) in the limit of vanishing dissipation, we shall see in due course that this is a "singular" limit, in the sense that the processes of this group exhibit very different features.

There is a similarity between the fundamental (phenomenological) laws governing processes of the same group. For example, Fourier's law of heat conduction, Fick's law of diffusion, and Darcy's law of liquid percolation through porous media are identical, up to renaming of the variables. Indeed, *Fourier's law* reads: The amount of heat flowing in an isotropic homogeneous thermally conductive body through a surface element $d\tilde{\sigma}$ in the direction of the normal $\mathbf{n}$ to $d\tilde{\sigma}$ in time $d\tilde{t}$ is

$$d\tilde{q} = -\tilde{\lambda}\frac{\partial}{\partial n}\tilde{T}d\tilde{\sigma}d\tilde{t} \tag{1.2}$$

where $\tilde{T}$ is the temperature and the minus sign indicates that the heat is flowing in the direction of decreasing temperature, so that the coefficient $\tilde{\lambda}$ of thermal conductivity may be assumed positive.

Now *Fick's law* reads as follows: The mass of a solute transferred by diffusion in an isotropic solution through a surface element $d\sigma$ in the direction of the normal $\mathbf{n}$ to $d\tilde{\sigma}$ in time $d\tilde{t}$ is

$$d\tilde{q} = -\tilde{D}\frac{\partial}{\partial n}\tilde{C}d\tilde{\sigma}d\tilde{t} \tag{1.3}$$

where $\tilde{C}$ is the solute concentration.

Finally, *Darcy's law* reads: The mass of liquid percolating through the pore space of a homogeneous porous medium through a surface element $d\tilde{\sigma}$ in direction of the normal $\mathbf{n}$ to $d\tilde{\sigma}$ in time $d\tilde{t}$ is

$$d\tilde{q} = -\tilde{K}\frac{\partial}{\partial n}\tilde{p}d\tilde{\sigma}d\tilde{t} \tag{1.4}$$

where $\tilde{p}$ is the pore pressure and $\tilde{K}$ the percolation coefficient. The tilde "$\tilde{\ }$" indicates that variables are dimensional.

All these phenomenological laws are of the same form; written in terms of dimensionless variables, they are indistinguishable. It is this possibility of simultaneously describing processes of a different physical nature, but belonging to the same group, that makes mathematical physics a universal language of the continuum, a connecting link between different disciplines of physics, chemistry, biology, and so on. The interrelation between the various properties of partial differential equations (PDEs) and

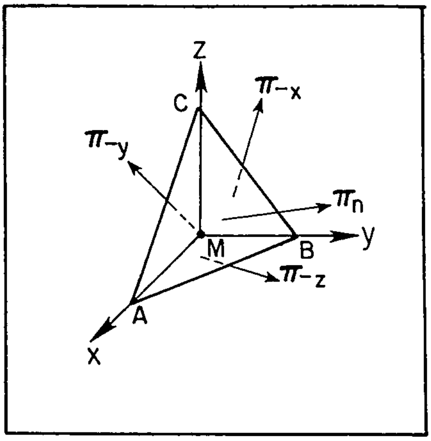

Figure 1.1. Sketch of stresses on the tetrahedron $MABC$.

their natural prototypes is very profound and helpful for research. Quite frequently, previously unknown mathematical phenomena are discovered by looking for the explanation of a natural phenomenon and vice versa, a natural phenomenon may be predicted by analyzing the properties of the corresponding mathematical object.

Before going on to a systematic study of the classical methods of mathematical physics, we shall present some preliminary information about the natural systems that will be referred to later. This includes some basic facts from thermodynamics, continuum mechanics, electrodynamics, and chemical kinetics. For a systematic presentation of these topics the reader is referred to the relevant specialized texts.

2. Basic concepts of continuum mechanics [47]

A. Mass (volume) forces and surface forces (stresses)

Mass (or volume) forces, such as forces of gravity and inertia (e.g., centrifugal force), are additive functions of the mass (volume) of a material body and are thus characterized by their densities per unit mass (or volume).

Surface forces (e.g., forces due to molecular interaction) are additive functions of the body's surface area and are thus characterized by their surface density (stress). Consider some domain D in a continuous medium. Let Σ be a two-sided smooth surface dividing D into two parts D_i and D_e, and let the normal $\mathbf{n}$ to Σ be directed into D_e. The surface force with which D_e attracts D_i, called the *stress*, is characterized by its density $\mathbf{\Pi}_n(P)$ (force per unit area of Σ at point P).

Let (x_1, x_2, x_3) be an orthogonal cartesian coordinate system and $\mathbf{i}_k$, $k = 1, 2, 3$, the basis vectors of the system (i.e., unit vectors directed along the x_k axis). Let us express $\mathbf{\Pi}_n$ in terms of $\mathbf{\Pi}_k$, $k = 1, 2, 3$. ($\mathbf{\Pi}_k$ is a stress on the surface with a normal $\mathbf{i}_k$). Imagine an infinitesimal tetrahedron $MABC$ cut out of the medium (see Figure 1.1). Let $d\sigma_1$, $d\sigma_2$, $d\sigma_3$, and

$d\sigma_n$ denote the surface areas of the faces MBC, MAC, MAB, and ABC, respectively. If h_n is the altitude of the tetrahedron dropped from its vertex M, $\alpha_1, \alpha_2, \alpha_3$ are the direction cosines of $\mathbf{n}$ and $d\omega$ is the volume of $MABC$, then

$$d\sigma_i = \alpha_i \cdot d\sigma_n, \quad d\omega = h_n \cdot d\sigma_n/3, \quad i = 1,2,3 \tag{1.5}$$

The surface forces acting on the tetrahedron at its faces MBC, MAC, MAB, and ABC are, respectively,

$$-\mathbf{\Pi}_1 d\sigma_1, \ -\mathbf{\Pi}_2 d\sigma_2, \ -\mathbf{\Pi}_3 d\sigma_3, \ \mathbf{\Pi}_n d\sigma_n \tag{1.6}$$

The mass force F acting on the tetrahedron is

$$\mathbf{F} = \mathbf{f}\rho d\omega \tag{1.7}$$

where $\mathbf{f}$ is the mass force density and ρ the density of the medium. Finally, the force of inertia is

$$\mathbf{F} = -\rho \frac{d}{dt}\mathbf{v} d\omega \tag{1.8}$$

Thus, by Newton's second law applied to the tetrahedron $MABC$, formulas (1.5)–(1.8) yield

$$\mathbf{\Pi}_n - \sum_{i=1}^{3} \alpha_i \mathbf{\Pi}_i - h_n \rho \left(\mathbf{f} + \frac{d}{dt}\mathbf{v} \right) = 0 \tag{1.9}$$

Hence, going to the limit of $h_n \downarrow 0$ we obtain condition of local mechanical equilibrium

$$\mathbf{\Pi}_n = \sum_{i=1}^{3} \alpha_i \mathbf{\Pi}_i \tag{1.10}$$

Now let

$$\Pi_{nj}, \ \Pi_{ij}, \quad i,j = 1,2,3 \tag{1.11}$$

be the jth components of the vectors $\mathbf{\Pi}_n$ and $\mathbf{\Pi}_i$, respectively. Then by (1.10),

$$\Pi_{nj} = \sum_{i=1}^{3} \alpha_i \cdot \Pi_{ij}, \quad j = 1,2,3 \tag{1.12}$$

According to (1.12), the stress on an arbitrarily oriented surface with normal $\mathbf{n}(\alpha_1, \alpha_2, \alpha_3)$ is determined by a tensor

$$\tilde{\mathbf{\Pi}} = (\Pi_{ij}) \tag{1.13}$$

known as the *stress tensor*. Reasoning as in (1.5)–(1.10) for the momenta of the surface forces on the tetrahedron $MABC$, one sees that the tensor $\tilde{\mathbf{\Pi}}$ is symmetric, that is

$$\Pi_{ij} = \Pi_{ji}, \quad i,j = 1,2,3 \tag{1.14}$$

The diagonal components Π_{ii} of $\tilde{\mathbf{\Pi}}$ are known as *normal* stresses, the nondiagonal ones *tangential* or *shear* stresses.

Define $\tilde{p}$ to be minus one third of the trace of the stress tensor

$$\tilde{p} = -\frac{1}{3}\sum_{i=1}^{3}\Pi_{ii} \tag{1.15}$$

This quantity is known as the *pressure.* Let $\tilde{\mathbf{I}}$ be the unit tensor, that is, the tensor whose components are

$$I_{jk} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \ \ j = 1,2,3 \\ 0 & \text{if } i \neq j, \ \ i,j = 1,2,3 \end{cases} \tag{1.16}$$

Finally, let $\tilde{\tau}$ be the deviator of the tensor $\tilde{\mathbf{\Pi}}$:

$$\tilde{\tau} = \tilde{\mathbf{\Pi}} + \tilde{p}\tilde{\mathbf{I}} \tag{1.17}$$

A continuum for which $\tilde{\tau}$ vanishes identically (admits no shear stresses at any point) and for which the only normal stress is pressure (with a minus sign) is occasionally called *perfect* (*ideal*).

B. Deformation of continuous medium. Strain tensor and strain velocity tensor

The deformation of a continuum is described as follows. Let (x_1, x_2, x_3) be a rectangular cartesian coordinate system, $p = (x_1, x_2, x_3)$ some material point in the medium, and $q = (\xi_1, \xi_2, \xi_3)$ the position of the same material point in the deformed state. Let us assume that the Jacobian

$$\frac{D(x_1, x_2, x_3)}{D(\xi_1, \xi_2, \xi_3)} \neq 0 \tag{1.18}$$

so that the x_i may be considered as implicit functions of ξ_j, $i,j = 1,2,3$.Consider a line element

$$\mathbf{ds} = \sum_{i=1}^{3} dx_i \mathbf{e}_i \tag{1.19}$$

in the undeformed body, which transforms into a line element $\mathbf{d}\tilde{\mathbf{s}}$ as result of the deformation:

$$\mathbf{d}\tilde{\mathbf{s}} = \sum_{i=1}^{3} d\xi_i \mathbf{e}_i \tag{1.20}$$

The vector

$$\mathbf{r}_{pq} = \sum_{i=1}^{3} V_i \mathbf{e}_i \tag{1.21}$$

where

$$V_i = \xi_i - x_i, \quad i = 1,2,3 \tag{1.22}$$

is known as the *displacement vector*. Let ds and $d\tilde{s}$ be the absolute values of $\mathbf{ds}$ and $\mathbf{d\tilde{s}}$, respectively. The relative extension of $\mathbf{ds}$ is

$$e = \frac{d\tilde{s}}{ds} - 1. \tag{1.23}$$

By (1.19)–(1.22),

$$d\xi_1 = \left(1 + \frac{\partial}{\partial x_1}V_1\right)dx_1 + \frac{\partial}{\partial x_2}V_1 dx_2 + \frac{\partial}{\partial x_3}V_1 dx_3 \tag{1.24}$$

$$d\xi_2 = \frac{\partial}{\partial x_1}V_2 dx_1 + \left(1 + \frac{\partial}{\partial x_2}V_2\right)dx_2 + \frac{\partial}{\partial x_3}V_2 dx_3 \tag{1.25}$$

$$d\xi_3 = \frac{\partial}{\partial x_1}V_3 dx_1 + \frac{\partial}{\partial x_2}V_3 dx_2 + \left(1 + \frac{\partial}{\partial x_3}V_3\right)dx_3 \tag{1.26}$$

Let l_i and $\bar{l}_i$, $i = 1, 2, 3$, be the direction cosines of the line elements $\mathbf{ds}$ and $\mathbf{d\tilde{s}}$, respectively, so that

$$ds = \sum_{i=1}^{3} l_i dx_i, \qquad d\tilde{s} = \sum_{i=1}^{3} \tilde{l}_i d\xi_i \tag{1.27}$$

Since

$$\sum_{i=1}^{3} l_i^2 = 1, \qquad \sum_{i=1}^{3} \tilde{l}_i^2 = 1 \tag{1.28}$$

equalities (1.24)–(1.26) imply that

$$\left(\frac{d\tilde{s}}{ds}\right)^2 - 1 = e(e+2) = 2F(l_1, l_2, l_3) \tag{1.29}$$

where

$$F(l_1, l_2, l_3) = \sum_{i=1}^{3}\sum_{j=1}^{3} \varepsilon_{ij} l_i l_j \tag{1.30}$$

Here

$$\varepsilon_{ij} = e_{ij} + e_{ij}^* \tag{1.31}$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right) \tag{1.32}$$

$$e_{i_j}^* = \frac{1}{2}\frac{\partial V}{\partial x_i}\frac{\partial V}{\partial x_j} \tag{1.33}$$

Thus the quadratic form F consists of two parts: one part that is linear in the derivatives of the displacement vector,

$$f(l_1, l_2, l_3) = \sum_{i=1}^{3}\sum_{j=1}^{3} e_{ij} l_i l_j \tag{1.34}$$

and a bilinear part,

$$f^*(l_1, l_2, l_3) = \sum_{i=1}^{3}\sum_{j=1}^{3} e^*_{ij} l_i l_j \tag{1.35}$$

The quadratic forms F and f generate two tensors

$$\tilde{\varepsilon} = (\varepsilon_{ij}) \tag{1.36}$$

$$\tilde{\mathrm{e}} = (e_{ij}) \tag{1.37}$$

$\tilde{\varepsilon}$ is known as the tensor of finite deformations, and $\tilde{e}$ as the tensor of small deformations or the strain tensor.[1] If the process under consideration is time dependent, the tensors

$$\tilde{\varepsilon}' = (\dot{\varepsilon}_{ij}), \quad \tilde{\mathrm{e}}' = (\dot{e}_{ij}) \tag{1.38}$$

are known, respectively, as the rate of finite deformation and rate of strain tensors. (The dot designates partial differentiation with respect to time.)

C. Volumetric dilatation

Volumetric dilatation is defined as

$$\Theta = \frac{\omega_1 - \omega}{\omega} \tag{1.39}$$

where ω and ω_1 are the specific volumes of the medium before and after deformation, evaluated at the same material point. Let ρ_1 and ρ be the densities of the medium in the deformed and undeformed states, respectively. Then

$$\rho = \frac{1}{\omega} \Rightarrow \Theta = \frac{\rho - \rho_1}{\rho_1} \tag{1.40}$$

Let M be the mass of some fixed set of material particles, occupying an arbitrarily chosen simply connected region D at some time, so that

$$M = \int_D \rho(q)dx \tag{1.41}$$

where dx denotes the volume element

$$dx = dx_1 dx_2 dx_3 \tag{1.42}$$

After deformation, the same set of material points occupies a deformed region D_1, so that

$$M = \int_{D_1} \rho_1 d\xi \tag{1.43}$$

where

$$\xi_i = x_i + V_i(x_1, x_2, x_3) \tag{1.44}$$

[1]In what follows we consider only small deformations. Accordingly, we use the term "strain tensor" only for small deformations.

and V_i are the components of the displacement vector.

We have

$$M = \int_{D_1} \rho_1 d\xi = \int_D \rho_1 \left| \frac{D(\xi_1, \xi_2, \xi_3)}{D(x_1, x_2, x_3)} \right| dx \tag{1.45}$$

where

$$\frac{D(\xi_1, \xi_2, \xi_3)}{D(x_1, x_2, x_3)} = \begin{vmatrix} 1 + \frac{\partial}{\partial x_1} V_1 & \frac{\partial}{\partial x_2} V_2 & \frac{\partial}{\partial x_3} V_1 \\ \frac{\partial}{\partial x_1} V_2 & 1 + \frac{\partial}{\partial x_2} V_2 & \frac{\partial}{\partial x_3} V_2 \\ \frac{\partial}{\partial x_1} V_3 & \frac{\partial}{\partial x_2} V_3 & 1 + \frac{\partial}{\partial x_3} V_3 \end{vmatrix} = 1 + \operatorname{div} V + A \tag{1.46}$$

with A containing all quadratic terms in the components of the displacement vector $\mathbf{V}$. Since only small deformations are considered, the Jacobian (1.46) is positive. Therefore, there is no need for absolute-value signs in (1.45).

Equalities 1.41 and 1.45 yield

$$\Theta = \int_D [\rho - \rho_1[1 + \operatorname{div} V + A]]\, dx \tag{1.47}$$

Since D is an arbitrary simply connected region and we are assuming that the integrand is continuous, it follows from (1.47) that

$$\rho - \rho_1[1 + \operatorname{div} V + A] = 0 \tag{1.48}$$

or, up to highest order terms in V,

$$\Theta \overset{\text{def}}{=} \frac{\rho - \rho_1}{\rho_1} = \operatorname{div} V \tag{1.49}$$

D. *Substantial (material) derivatives and continuity equation*

Assume that the deformation takes place during a time dt, so that

$$\mathbf{V} = \mathbf{v} dt \tag{1.50}$$

where $\mathbf{v}$ is the velocity of small deformations. Then

$$\rho_1 = \rho + \sum_{i=1}^{3} \frac{\partial \rho}{\partial x_i} v_i dt = \rho + \mathbf{v} \cdot \operatorname{grad} \rho dt \tag{1.51}$$

which implies

$$\Theta = -\frac{\frac{d}{dt}\rho dt}{\rho + \frac{d}{dt}\rho dt} \approx -\frac{1}{\rho}\frac{d}{dt}\rho dt \tag{1.52}$$

Comparing (1.52) with (1.49), we obtain

$$\frac{d}{dt}\rho + \rho \operatorname{div} \mathbf{v} = 0 \tag{1.53}$$

On the other hand, ρ_1 is the density of the medium at the point $(\xi_1, \xi_2, \xi_3, t + dt)$,

$$\rho_1 = \rho(\xi_1, \xi_2, \xi_3, t + dt) \tag{1.54}$$

so that

$$\frac{d}{dt}\rho = \frac{\partial}{\partial t}\rho + \sum_{i=1}^{3} \frac{\partial \rho}{\partial x_i} v_i \equiv \frac{\partial}{\partial t}\rho + \mathbf{v} \cdot \operatorname{grad} \rho \tag{1.55}$$

The operator

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \operatorname{grad} \tag{1.56}$$

is called the *substantial* (or *material*) *differentiation* operator; equation (1.53), that is, the local form of the law of mass conservation, is known as the *equation of continuity*[2]

E. Rheological state of continuous medium

The rheological state of a continuous medium is determined by the relationship between the stress tensor and the strain and rate of strain tensors. In the general case,

$$\tilde{\mathbf{\Pi}} = F(\tilde{\varepsilon}, \tilde{\varepsilon}') \tag{1.57}$$

The rheological equation of state of a linearly elastic isotropic body is given by a phenomenological law – Hook's law – which, as formulated by Lamé, reads

$$\Pi_{ij} = \lambda \operatorname{div} \mathbf{V} \delta_{ij} + 2\mu e_{ij} \tag{1.58}$$

where δ_{ij} is the Kronecker delta[3] and λ and μ are known as Lamé coefficients.

Similarly, the rheological equation of state of a real (Newtonian viscous) liquid is given by the phenomenological Navier law, in the form

$$\Pi_{ij} = -\left(p - \left(\frac{2}{3} \cdot \mu - \mu'\right) \operatorname{div} \mathbf{v}\right) \delta_{ij} + 2\mu \dot{e}_{ij} \tag{1.59}$$

If $\tilde{\tau}$ is the deviator of the stress tensor and $-\tilde{p}$ one third of the trace, then it follows from (1.15), (1.37), and (1.32) that

$$p = \tilde{p} - \mu' \operatorname{div} \mathbf{v} \tag{1.60}$$

The quantity p defined by (1.60) is the pressure in the liquid in a state of local thermodynamic equilibrium. If the liquid is incompressible, or if its

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[2]The equation of continuity will be derived again, in a different way, in Section 3 of Chapter 2.

[3]Recall that the Kronecker delta is defined by $\delta_{ii} = 1$, $\delta_{ij} = 0$, $i, j = 1, 2, 3$, $i \neq j$.

compressibility is small enough, the pressure p is minus one third the trace of the stress tensor. The coefficients μ and μ' are known, respectively, as the coefficients of dynamic (or Newtonian, or shear) viscosity and volume viscosity. In most hydrodynamic situations, the effect of volume viscosity is negligible. It must be taken into account, however, in the case of a compressible medium with a long relaxation process, for example, a gas with a slow chemical reaction.

The rheological equation of state for an inviscid (perfect) liquid is

$$\tilde{\mathbf{\Pi}} = -p\tilde{\mathbf{I}} \tag{1.61}$$

that is, there are no shear stresses in an inviscid liquid.

F. Kinematics of fluid motion. Inviscid fluid flow

The following definitions and theorems are basic for the kinematics of any liquid.

Definition 1.2.1. Let $\mathbf{v}$ be the velocity vector of fluid motion. The motion of the fluid is said to be *rotational*, if there exists a vector $\mathbf{\Omega}$, called the *vector velocity potential*, such that

$$\mathbf{v} = \operatorname{rot} \mathbf{\Omega} \tag{1.62}$$

Obviously, the velocity of rotational motion is solenoidal, in the sense that

$$\operatorname{div} \mathbf{v} = \operatorname{div} \operatorname{rot} \mathbf{\Omega} \equiv 0 \tag{1.63}$$

Definition 1.2.2. The motion of a fluid motion is said to be *potential* if there exists a scalar φ, called the *velocity potential*, such that

$$\mathbf{v} = \operatorname{grad} \varphi \tag{1.64}$$

Recall that any continuously differentiable vector function $\mathbf{v}$ may be represented as the sum of the rotor of a vector potential $\mathbf{\Omega}$ and the gradient of a scalar potential φ; that is, in general,

$$\mathbf{v} = \operatorname{rot} \mathbf{\Omega} + \operatorname{grad} \varphi \tag{1.65}$$

Definition 1.2.3. Let $\mathbf{A}(p, t)$ be a vector field. A curve Γ is called a *vector line* if it is tangent to $\mathbf{A}$ at each of its points.

In particular, if $\mathbf{A}(p, t)$ is a field of vorticity $\boldsymbol{\omega}$, defined as

$$\boldsymbol{\omega} = \operatorname{rot} \mathbf{v} \tag{1.66}$$

then the vector line of $\boldsymbol{\omega}$ is called a *vortex line.*

Definition 1.2.4. Let $L \subset \mathbb{R}_3$ be a closed curve and let S be the surface composed of the vector lines Γ of $\mathbf{A}$, passing through L. Then S is called