Chapter 1

Prelude — Quadratic Polynomials and Quadratic Forms

Classical invariant theory is the study of the intrinsic properties of polynomials. By "intrinsic", we mean those properties which are unaffected by a change of variables and are hence purely geometric, untied to the explicit coordinate system in use at the time. Thus, properties such as factorizability and multiplicities of roots, as well as their geometrical configurations, are intrinsic, whereas the explicit values of the roots and the particular coefficients of the polynomial are not. The study of invariants is closely tied to the problem of equivalence — when can one polynomial be transformed into another by a suitable change of coordinates — and the associated canonical form problem — to find a system of coordinates in which the polynomial takes on a particularly simple form. The solution to these intimately related problems, and much more, are governed by the invariants, and so the first goal of classical invariant theory is to determine the fundamental invariants. With a sufficient number of invariants in hand, one can effectively solve the equivalence, and canonical form problems, and, at least in principle, completely characterize the underlying geometry of a given polynomial.

All of these issues are already apparent in the very simplest example — that of a quadratic polynomial in a single variable. This case served as the original catalyst for Boole and Cayley's pioneering work in the subject, [**24**, **36**], and can be effectively used as a simple (i.e., just high school algebra is required) concrete example that will motivate our study of the subject. We shall devote this introductory chapter to the elementary theory of quadratic polynomials in a single variable, together with homogeneous quadratic forms in two variables. Readers who are unimpressed with such relative trivialities are advised to proceed directly to the true beginning of our text in Chapter 2.

Quadratic Polynomials

Consider a quadratic polynomial in a single variable p:

$$Q(p) = ap^2 + 2bp + c. \tag{1.1}$$

Before addressing the question of what constitutes an invariant in this context, we begin our analysis with the elementary problem of determining a canonical form for the polynomial Q. In other words, we are trying to make Q as simple as possible by use of a suitable change of variable. As long as $a \neq 0$, the two roots of Q are, of course, given by the justly famous *quadratic formula*

$$p_+ = \frac{-b + \sqrt{-\Delta}}{a}, \qquad p_- = \frac{-b - \sqrt{-\Delta}}{a}, \tag{1.2}$$

where

$$\Delta = ac - b^2 \tag{1.3}$$

is the familiar *discriminant*† of Q. The existence of the two roots implies that we can factor

$$Q(p) = a(p - p_+)(p - p_-) \tag{1.4}$$

into two linear, possibly complex-valued, factors.

At this point, we need to be a bit more specific as to whether we are dealing with real or complex polynomials. Let us first concentrate on the slightly simpler complex version. The most obvious changes of variables preserving the class of quadratic polynomials are the affine transformations

$$\bar{p} = \alpha p + \beta, \tag{1.5}$$

for complex constants $\alpha \neq 0$ and β. Here α represents a (complex) scaling transformation,‡ and β a complex translation. The transformation (1.5) maps the original quadratic polynomial $Q(p)$ to a new quadratic polynomial $\overline{Q}(\bar{p})$, which is constructed so that

$$\overline{Q}(\bar{p}) = \overline{Q}(\alpha p + \beta) = Q(p). \tag{1.6}$$

In particular, if p_0 is a root of $Q(p)$, then $\bar{p}_0 = \alpha p_0 + \beta$ will be a root of $\overline{Q}(\bar{p})$. For example, if $Q(p) = p^2 - 1$, and we apply the transformation

† The sign chosen for the discriminant is in accordance with later generalizations.

‡ If we write $\alpha = re^{i\theta}$, then the modulus r will act by scaling, whereas the exponential $e^{i\theta}$ will induce a rotation in the complex p-plane; see p. 46.

$\bar{p} = 2p - 1$, then $\overline{Q}(\bar{p}) = \frac{1}{4}\bar{p}^2 + \frac{1}{2}\bar{p} - \frac{3}{4}$. The roots $p_\pm = \pm 1$ of Q are mapped to the roots $\bar{p}_+ = 1$ and $\bar{p}_- = -3$ of $\overline{Q}$.

For a general complex quadratic polynomial, there are only two cases to consider. If its discriminant is nonzero, $\Delta \neq 0$, then the roots of Q are distinct. We can translate one root, say, p_-, to be zero and then scale so that the second root takes the value 1. Thus, by a suitable choice of α and β we can arrange that $\overline{Q}$ has its roots at 0 and 1. Consequently, under complex affine transformations (1.5), every quadratic polynomial with distinct roots can be placed in the canonical form $\overline{Q}(\bar{p}) = k(\bar{p}^2 - \bar{p})$ for some $k \in \mathbb{C}$.

Exercise 1.1. Find the explicit formulas for α, β that will reduce a quadratic polynomial Q to its canonical form. Is the residual coefficient k uniquely determined? Determine the formula(e) for k in terms of the original coefficients of Q.

Exercise 1.2. An alternative canonical form for such quadratics is $\widetilde{Q}(\tilde{p}) = \tilde{k}\,(\tilde{p}^2 + 1)$. Do the same exercise for this canonical form, and describe what is happening to the roots of Q.

On the other hand, if the discriminant of Q vanishes, so $ac = b^2$, then Q has a single double root p_0 and so factors as a perfect square: $Q(p) = a(p - p_0)^2$. Clearly this property is intrinsic — it cannot be altered by any change of coordinates. We can translate the double root to the origin, reducing Q to a multiple of the polynomial $\tilde{p}^2$, and then scale the coordinate $\tilde{p}$ to reduce the multiple to unity, leading to a canonical form, $\overline{Q} = \bar{p}^2$, for a quadratic polynomial with a double root.

We are not quite finished, since we began by assuming that the leading coefficient $a \neq 0$. If $a = 0$, but $b \neq 0$, then Q reduces to a linear polynomial with a single root, $p_0 = -c/b$. We can, as in the preceding case, translate this root to 0 and then scale, producing the canonical form $\overline{Q} = \bar{p}$ in this case. If $b = 0$ also, then Q is a constant, and, from the viewpoint of affine transformations (1.5), there is nothing that can be done. Thus, we have constructed a complete list of canonical forms for quadratic polynomials, under complex affine changes of coordinates. Note particularly that the discriminant Δ and the leading coefficient a play distinguished roles in the classification.

Exercise 1.3. Suppose Q and $\overline{Q}$ are related by an affine change of variables (1.5). Determine how their discriminants and leading coefficients are related.

Affine Canonical Forms for Complex Quadratic Polynomials			
I.	$k(p^2+1)$	$\Delta \neq 0,\ a \neq 0$	distinct roots
II.	p^2	$\Delta = 0,\ a \neq 0$	double root
III.	p	$a = 0,\ b \neq 0$	linear
IV.	c	$a = b = 0$	constant

The case of real polynomials under real affine changes of coordinates is similar, but there are a few more cases to consider. First, note that the roots (1.2) of a real quadratic polynomial are either both real or form a complex conjugate pair, depending on the sign of the discriminant. If Q has complex conjugate roots, meaning that its discriminant is positive, then it can never be mapped, under a real change of variables, to a quadratic polynomial with real roots, and so our complex canonical form is not as universally valid in this case. However, if the two roots are $p_\pm = r \pm is$, then a translation by $\beta = -r$ will move them onto the imaginary axis; this may be followed by a scaling to place them at $\pm i$. Thus, the canonical form in this case is $k(p^2+1)$. On the other hand, if the discriminant is negative, then Q has two distinct real roots, which can be moved to ± 1, leading to the alternative canonical form $k(p^2-1)$. The remaining cases are as in the complex version, since a double root of a real quadratic polynomial is necessarily real. We therefore deduce the corresponding table of real canonical forms.

Affine Canonical Forms for Real Quadratic Polynomials			
Ia.	$k(p^2+1)$	$\Delta > 0,\ a \neq 0$	complex conjugate roots
Ib.	$k(p^2-1)$	$\Delta < 0,\ a \neq 0$	distinct real roots
II.	p^2	$\Delta = 0,\ a \neq 0$	double root
III.	p	$a = 0,\ b \neq 0$	single root
IV.	c	$a = b = 0$	constant

Exercise 1.4. Determine the possible canonical forms for a complex cubic polynomial $Q(p) = ap^3 + bp^2 + cp + d$ under affine changes of coordinates. *Hint*: What are the possible root configurations?

Quadratic Forms and Projective Transformations

While affine changes of coordinates are immediately evident, they do not form the most general class that preserves the space of polynomials. In order to motivate a further extension, we begin by explaining the connection between homogeneous and inhomogeneous polynomials. Instead of the inhomogeneous polynomial (1.1) in a single variable, we consider the homogeneous quadratic polynomial

$$Q(x,y) = ax^2 + 2bxy + cy^2, \tag{1.7}$$

in two variables x, y, known classically as a *quadratic form*. Clearly we can recover the inhomogeneous quadratic polynomial $Q(p)$ from the associated quadratic form $Q(x,y)$ by setting $p = x$ and $y = 1$, so that $Q(p) = Q(p,1)$. On the other hand, the homogeneous version (1.7) can be directly constructed from $Q(p)$ according to the basic formula

$$Q(x,y) = y^2\, Q\left(\frac{x}{y}\right). \tag{1.8}$$

An affine change of coordinates (1.5) will induce a linear transformation mapping the quadratic form $Q(x,y)$ associated with $Q(p)$ to the quadratic form $\overline{Q}(\bar{x},\bar{y})$ associated with $\overline{Q}(\bar{p})$, as defined in (1.6). Clearly, the upper triangular linear transformation

$$\bar{x} = \alpha x + \beta y, \qquad \bar{y} = y, \qquad \alpha \neq 0, \tag{1.9}$$

will have the desired effect on the quadratic forms:

$$\overline{Q}(\bar{x},\bar{y}) = \overline{Q}(\alpha x + \beta y, y) = Q(x,y).$$

We conclude that the theory of inhomogeneous quadratic polynomials under affine coordinate changes is isomorphic to the theory of quadratic forms under linear transformations of the form (1.9).

Now, a crucial observation is that the class of quadratic forms is preserved under a much wider collection of transformation rules. Namely, *any* invertible linear change of variables

$$\bar{x} = \alpha x + \beta y, \qquad \bar{y} = \gamma x + \delta y, \qquad \alpha\delta - \beta\gamma \neq 0, \tag{1.10}$$

will map a homogeneous polynomial in x and y to a homogeneous polynomial in $\bar{x}$ and $\bar{y}$ according to

$$\overline{Q}(\bar{x},\bar{y}) = \overline{Q}(\alpha x + \beta y, \gamma x + \delta y) = Q(x,y). \tag{1.11}$$

The coefficients of the transformed polynomial

$$\overline{Q}(\bar{x},\bar{y}) = \bar{a}\,\bar{x}^2 + 2\bar{b}\,\bar{x}\bar{y} + \bar{c}\,\bar{y}^2$$

are constructed from those of the original polynomial (1.7) according to the explicit formulae

$$\begin{aligned} a &= \alpha^2\,\bar a + 2\alpha\gamma\,\bar b + \gamma^2\,\bar c, \\ b &= \alpha\beta\,\bar a + (\alpha\delta + \beta\gamma)\,\bar b + \gamma\delta\,\bar c, \\ c &= \beta^2\,\bar a + 2\beta\delta\,\bar b + \delta^2\,\bar c. \end{aligned} \tag{1.12}$$

Remarkably, the discriminant of the transformed polynomial is directly related to that of the original quadratic form — a straightforward computation verifies that they agree up to the square of the determinant of the coefficient matrix $A = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ for the linear transformation (1.10):

$$\Delta = ac - b^2 = (\alpha\delta - \beta\gamma)^2\,(\bar a\bar c - \bar b^2) = (\alpha\delta - \beta\gamma)^2\,\overline{\Delta}. \tag{1.13}$$

The transformation rule (1.13) expresses the underlying invariance of the discriminant of a quadratic polynomial and provides the simplest example of an invariant (in the sense of classical invariant theory).

Remark: A linear transformation (1.10) is called *unimodular* if it has unit determinant $\alpha\delta - \beta\gamma = 1$ and hence preserves planar areas. For the restricted class of unimodular transformations, the discriminant is a bona fide invariant: $\overline{\Delta} = \Delta$.

What is the effect of a general linear transformation on the original inhomogeneous polynomial? For this purpose, it helps to refer back to the formula (1.8) relating the inhomogeneous polynomial and its homogeneous counterpart. Specifically, the inhomogeneous or projective variable p is identified with the ratio of the homogeneous variables, so $p = x/y$. Therefore, the effect of the linear transformation (1.10) is to transform the projective variable p according to the *linear fractional* or *Möbius* or *projective transformation*

$$\bar p = \frac{\alpha p + \beta}{\gamma p + \delta}, \qquad \alpha\delta - \beta\gamma \neq 0. \tag{1.14}$$

Thus, each invertible 2×2 coefficient matrix $A = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ induces an invertible transformation mapping p to $\bar p$, which is defined everywhere except, when $\gamma \neq 0$, at the singular point $p = -\delta/\gamma$. Such transformations lie at the heart of projective geometry and are of fundamental importance, not just in invariant theory, but in a wide range of classical and modern disciplines, including complex analysis and geometry, [**8**], number theory, [**79**], and hyperbolic geometry, [**20**]. Indeed, this rather simple construction has led to some of the most profound consequences in all of mathematics.

Exercise 1.5. Prove directly that the composition of two linear fractional transformations is again a linear fractional transformation, whose coefficients are obtained by multiplying the associated 2×2 coefficient matrices. In particular, the inverse of a linear fractional transformation is the linear fractional transformation determined by the inverse coefficient matrix.

Exercise 1.6. Show that two coefficient matrices A and $\widetilde{A}$ determine the same linear fractional transformation (1.14) if and only if they are scalar multiples of each other: $A = \lambda \widetilde{A}$. Thus, in the complex case, any linear fractional transformation (1.14) can be implemented by a unimodular coefficient matrix: $\det A = 1$. What is the unimodular linear transformation associated with the affine transformation (1.5)? Is this result valid in the real case?

How should a general linear fractional transformation act on an inhomogeneous quadratic polynomial (1.1)? We want to maintain the transformation rules (1.12) on the coefficients, so that the action will be the inhomogeneous counterpart to the linear action (1.11) on homogeneous quadratic forms. This requires that the quadratic polynomials $Q(p)$ and $\overline{Q}(\bar{p})$ are related according to the basic formula

$$Q(p) = (\gamma p + \delta)^2\, \overline{Q}(\bar{p}) = (\gamma p + \delta)^2\, \overline{Q}\left(\frac{\alpha p + \beta}{\gamma p + \delta}\right). \qquad (1.15)$$

Note that the additional factor $(\gamma p + \delta)^2$, known as the quadratic *multiplier*, is used to clear denominators so that the linear fractional transformation (1.14) will still map quadratic forms to quadratic forms. The reader might enjoy verifying that the transformation rules (1.15) does lead to exactly the same formulae (1.12) for the coefficients, and hence the discriminant continues to satisfy the basic invariance criterion (1.13). Note that, even though two coefficient matrices which are scalar multiples of each other determine the *same* linear fractional transformation (1.14), their action on quadratic polynomials (1.15) is *different* (unless $A = \pm \widetilde{A}$), owing to the effect of the multiplier.

Exercise 1.7. Show that the inversion $\bar{p} = 1/(p+1)$ maps the quadratic polynomial $Q(p) = p^2 - 1$ to the linear polynomial $\overline{Q}(\bar{p}) = -2\bar{p} + 1$. Thus projective transformations do not necessarily preserve the degree of a polynomial. Given a linear fractional transformation (1.14), determine which quadratic polynomials $Q(p)$ are mapped to linear polynomials. Which are mapped to constant polynomials?

Let us return to the canonical form problem for quadratic polynomials, now rearmed with the more general projective transformations. Clearly, by suitably combining the transformations and appealing to Exercise 1.5, we can begin by placing the quadratic polynomial in canonical affine form. Consider first the complex canonical form $Q = k(p^2 + 1)$. If we scale according to the coefficient matrix $A = \lambda \mathbb{1}$, where $\mathbb{1}$ is the 2×2 identity matrix and $\lambda^2 = k$, then we can normalize $Q \mapsto p^2 + 1$. Furthermore, the transformation $\bar{p} = (p - i)/(p + i)$ will map $p^2 + 1$ to the linear polynomial $\bar{p}$. Therefore, if $\Delta \neq 0$, and so $Q(p)$ either has two distinct roots or is a nonzero linear polynomial, then there is just one canonical form, namely $Q(p) = p$. On the other hand, if we take the affine canonical form $Q = p^2$, we can apply the inversion $\bar{p} = 1/p$ to map it to the constant polynomial $\overline{Q} = 1$; further, any other constant (nonzero) polynomial can, by applying a diagonal scaling matrix, be mapped to the constant 1. Thus, for complex quadratic polynomials under general linear fractional transformations, there are only three canonical forms: the first is p, which occurs when $\Delta \neq 0$; the second is 1, which occurs when $\Delta = 0$ but Q is not identically 0; and the last is the most trivial case, namely $Q \equiv 0$.

Canonical Forms for Complex Quadratic Polynomials

I.	p	$\Delta \neq 0$	distinct roots
II.	1	$\Delta = 0,\ Q \not\equiv 0$	double root
III.	0	$Q \equiv 0$	

Thus, under projective transformations, every complex quadratic polynomial is equivalent to a linear or constant polynomial. Since the action of linear fractional transformations on inhomogeneous quadratic polynomials mirrors that of linear transformations on homogeneous quadratic forms, each of our canonical forms has a homogeneous counterpart. We conclude that, under complex linear transformations, there are also three different canonical quadratic forms: first, xy, or, alternatively, $x^2 + y^2$; second, x^2; and, third, the trivial zero form 0.

In the real case, note that the transformation rules (1.3) for the discriminant imply that its sign is invariant: if $\Delta > 0$, say, then $\overline{\Delta} > 0$ also. Of course, this just means that one cannot map real roots to complex roots by a real projective transformation. Moreover, the sign of Q itself is also invariant; one cannot map a positive definite quadratic polynomial to an indefinite or negative definite one. Consequently, there are three different canonical forms with nonvanishing discriminant. The sign of Q also affects the classification of quadratics with vanishing discriminant.

Canonical Forms for Real Quadratic Polynomials

Ia.	p^2+1	$\Delta > 0,\ Q \geq 0$	complex roots
Ib.	$-p^2-1$	$\Delta > 0,\ Q \leq 0$	complex roots
Ic.	p	$\Delta < 0$	distinct real roots
IIa.	1	$\Delta = 0,\ Q \geq 0$	double root
IIb.	-1	$\Delta = 0,\ Q \leq 0$	double root
III.	0	$Q \equiv 0$	

The corresponding homogeneous canonical forms are the positive definite, x^2+y^2, negative definite, $-x^2-y^2$, and indefinite, which can be taken as either xy or x^2-y^2, all of which were complex-equivalent, followed by the degenerate cases x^2, $-x^2$, and 0.

Suppose we restrict to area-preserving transformations, with unimodular coefficient matrix: $\det A = 1$. In this case, the discriminant is strictly invariant, and hence we can no longer rescale to normalize it to be ± 1. Retracing the preceding arguments, we see that the only effect is to introduce an extra scaling factor into the list of canonical forms. Thus, for complex-valued quadratic polynomials under area-preserving changes of variables, the canonical forms having nonzero discriminant become a one-parameter family of linear forms kp. Note that the inversion $\bar{p} = -1/p$ will map kp to $-k\bar{p}$, both of which have discriminant $\Delta = k^2$, but otherwise one cannot transform between two different linear canonical forms. Therefore, a complete list of canonical forms for complex quadratic polynomials under unimodular linear fractional transformations consists of the linear forms kp, along with the constant forms 1 and 0. In the real case, one similarly finds two families of canonical forms, $k(p^2+1)$ and kp, which are distinguished by the sign of the discriminant. In the degenerate cases where $\Delta = 0$, the list of canonical forms remains the same as before.

Remark: A generic unimodular linear fractional transformation depends on three free parameters: α, β, and γ. Further, a quadratic polynomial (1.1) has three coefficients. Thus, one might expect that one could normalize all three coefficients via a suitable choice of the three parameters in the linear fractional transformation. The invariance of the discriminant proves that this naïve parameter count can be misleading. (See Chapter 8 for a more sophisticated and accurate version, which is based on the orbit dimensions.)

Exercise 1.8. Determine the canonical forms for complex-valued quadratic polynomials under the class of real linear (or linear fractional) transformations. In other words, the coefficients a, b, c in (1.7) or (1.1) are allowed to be complex, but the transformations (1.10) or (1.14) are restricted so that $\alpha, \beta, \gamma, \delta$ are all real.

This concludes our brief presentation of the admittedly elementary theory of quadratic polynomials in one complex or one real variable. Extensions to multi-dimensional quadratic forms are certainly of interest, and we shall briefly return to this topic in Chapters 3 and 10. However, our more immediate interest is in extending these basic considerations to higher degree polynomials in a single variable and/or homogeneous polynomials in two variables. In the classical literature, these are known as "binary forms". Their invariants, geometry, and canonical forms, under projective and/or linear transformations, constitute the heart and soul of the classical theory.