1
First-order equations

1.1 Introductory remarks

Differential equations are encountered as relations between independent variables, such as t, and an unknown function $x(t)$ of t and some of its derivatives $\frac{d^j x}{dt^j}$, possibly expressed as

$$F\left(t, x, \frac{dx}{dt}, \ldots, \frac{d^n x}{dt^n}\right) = 0 \tag{1.1}$$

on some t interval. We typically seek the unknown $x(t)$ as a scalar, vector, or a matrix, depending on the dimensionality of F.

If partial derivatives with respect to one or more independent variables are involved, we have a *partial differential equation*. We shall, however, primarily consider *ordinary differential equations*, which involve only ordinary derivatives and are frequently expressed as coupled systems

$$\frac{dy_i}{dt} = f_i(y_1(t), y_2(t), \ldots, y_n(t), t), \;\; i = 1, 2, \ldots, n$$

of n scalar equations or, more concisely, as a vector system

$$\frac{dy}{dt} = f(y(t), t) \tag{1.2}$$

for

$$y \equiv \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \; \frac{dy}{dt} = y' \equiv \begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{pmatrix}, \text{ and } f \equiv \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$

More generally, we might encounter a vector system

$$g\left(t, y, \frac{dy}{dt}\right) = 0 \tag{1.3}$$

for, say, n-vectors y, y', and g. When we can solve such a relation to obtain y' as a function of y and t, we say the resulting system $y' = f(y, t)$ is *in standard form*.

Example 1 The scalar differential equation $(y')^2 + y^2 + 1 = 0$ corresponds to two differential equations,

$$y' = \pm i\sqrt{y^2 + 1},$$

in standard form. Complex-valued solutions would result from both possibilities; indeed, the differential equation does not have real solutions. Moreover, the original equation might be allowed solutions that switch (or "chatter") from time to time between the two sign possibilities whenever $y^2 = -1$. We will, however, limit attention below to differential equations with real-valued differentiable solutions.

Example 2 The *differential-algebraic system*

$$\begin{cases} (y_1')^2 = y_1^2 + y_2^2, \\ y_2' = g(y_1, y_2, y_3), \\ y_3^2 + y_2^2 = 4, \end{cases}$$

consisting of two scalar differential equations and one scalar "algebraic" equation, is equivalent to the two-dimensional differential systems

$$\begin{cases} y_1' = \pm\sqrt{y_1^2 + y_2^2}, \\ y_2' = g\left(y_1, y_2, \pm\sqrt{4 - y_2^2}\right) \end{cases}$$

in standard form, whose real solutions are subject to the constraints

$$y_3 = \pm\sqrt{4 - y_2^2} \quad \text{and} \quad -2 \le y_2 \le 2.$$

Any solution

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

of the two-dimensional differential systems satisfying the bound for y_2 defines two possibilities for y_3. As this example suggests, there is plenty of flexibility in defining solutions, although we shall seldom be adventurous.

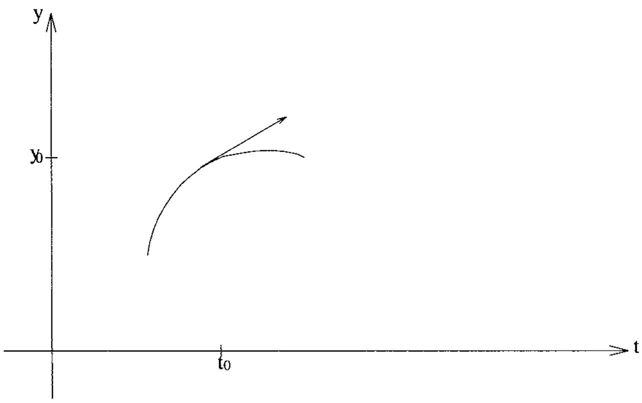

Figure 1. Tangent curve with slope $y'(t_0) = f(y_0, t_0)$ at (y_0, t_0) (scalar case).

If we are given the value $y(t_0)$ at some prescribed initial time t_0, it is natural to seek the solution $y(t)$ of the *initial value problem*

$$\frac{dy}{dt} = f(y(t), t), \quad y(t_0) = y_0 \tag{1.4}$$

for t values near t_0. We can readily approximate the solution as a short curve $y(t)$ through the given point (y_0, t_0) in (y, t) space with an initial *slope* $f(y_0, t_0)$. Note that (1.4) makes sense when y is a scalar, vector, or matrix unknown.

We would expect the solution $y(t)$ of (1.4) to be *unique* and to *exist* as long as $f(y(t), t)$ continues to be smoothly defined (and, in particular, remains bounded). Continuity of f and f_y will, indeed, suffice.

We naturally call the scalar equation

$$F\left(t, x, \frac{dx}{dt}, \ldots, \frac{d^n x}{dt^n}\right) = 0 \tag{1.1}$$

(for an unknown $x(t)$) an *nth order* differential equation, since the highest derivative involved is the nth (presuming actual dependence on the last argument of F). It can always be converted to an equivalent system of n first-order equations by, for example, setting

$$\begin{cases} y_1 = x \\ y_2 = \frac{dy_1}{dt} = x' \\ \vdots \\ y_{n-1} = \frac{dy_{n-2}}{dt} = x^{(n-2)} \\ y_n = \frac{dy_{n-1}}{dt} = x^{(n-1)}. \end{cases} \tag{1.5}$$

Since $\frac{dy_n}{dt} = x^{(n)}$, we can rewrite the original differential equation (1.1) as

$$F\left(t, y_1, y_2, \ldots, y_n, \frac{dy_n}{dt}\right) = 0.$$

Thus, a first-order vector system, equivalent to the nth-order scalar equation (1.1), for the vector variable y defined componentwise by (1.5), is

$$\begin{cases} \frac{dy_1}{dt} = y_2 \\ \frac{dy_2}{dt} = y_3 \\ \vdots \\ \frac{dy_{n-1}}{dt} = y_n \\ F\left(t, y_1, y_2, \ldots, y_n, \frac{dy_n}{dt}\right) = 0. \end{cases} \tag{1.6}$$

This system has a very special structure, suggesting that the general *first-order vector system*

$$g\left(t, y, \frac{dy}{dt}\right) = 0 \tag{1.7}$$

(for an arbitrary n vector g) is considerably more inclusive than higher-order scalar equations (1.1). Scalar equations, especially those of second order, nonetheless remain basic because they, for example, include Newton's law $\mathcal{F} = m\frac{d^2x}{dt^2}$ and many other fundamental laws of physics and engineering. Readers should become comfortable working with both first- and higher-order systems of ordinary differential equations.

Let us continue by considering two first-order examples from *population dynamics*.

Example 3 Suppose the population $N(t)$ of a given species (bacteria or elves) is not always zero and varies at a rate proportional to its current value, i.e.

$$\frac{dN}{dt} = rN \tag{1.8}$$

where r is some measured constant proportionality factor. Suppose the initial population $N(0) > 0$ is given. [If $N(0) = 0$, $\frac{dN}{dt}(0) = 0$ implies that $N(t) \equiv 0$ for all $t \geq 0$. (Without Adam and Eve, none of us would be here.)] Indeed, $N(t) \neq 0$ for all finite t (since $N(t_0) = 0$ and the change of independent variable $s = t - t_0$ yields $\frac{dN}{ds} = rN$ with $N = 0$ at $s = 0$.) Integrating $\frac{dN}{N} = r\,dt$ (since $N \neq 0$) implies that $\ln\left(\frac{N(t)}{N(0)}\right) = rt$, so

$$N(t) = e^{rt}N(0). \tag{1.9}$$

Thus,

$N(t)$ remains constant if $r = 0$;

$N(t)$ increases exponentially with t if $r > 0$;

and

$N(t)$ decreases monotonically, tending to zero as $t \to \infty$, if $r < 0$.

(Demographers: Is it realistic or relevant to let t become unbounded?)

Example 4 Worrying about a limited food supply and a constantly growing population, we might replace any positive proportionality constant r in the preceding *Malthusian* model (1.8) by a more biologically motivated *logistic function* $f(N)$, which will ultimately become negative for large values of N. Taking $f(N) = r\left(1 - \frac{N}{K}\right)$, for simplicity, we obtain

$$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)N \tag{1.10}$$

for a constant saturation level (or *carrying capacity*) $K > 0$ and an *intrinsic growth rate* $r > 0$. (Our preceding model thereby corresponds to the limiting possibility $K = \infty$.) Quadratic differential equations like (1.10) are usually called *Riccati equations*. They will be considered further below, because, although they are not simple, they are explicitly solvable.

We are able to determine the qualitative behavior of the population $N(t)$ as t varies without explicitly solving the differential equation. Simply note that

$$\frac{dN}{dt} > 0 \text{ for } 0 < N < K,$$

whereas

$$\frac{dN}{dt} < 0 \text{ for } K < N$$

and N remains constant at $N = 0$ and $N = K$. This implies that

$$\begin{cases} N(t) \equiv 0 & \text{if } N(0) = 0, \\ N(t) \uparrow K & \text{if } 0 < N(0) < K, \text{ where the } \textit{steady-state } K \text{ is} \\ & \text{attained from below as } t \to \infty, \\ N(t) \equiv K & \text{if } N(0) = K, \\ \text{and} & \\ N(t) \downarrow K & \text{if } N(0) > K \end{cases} \tag{1.11}$$

Thus, the solution $N(t)$ of the initial value problem is always a monotonic function of the independent variable $t \geq 0$, and the limit K is attained for $N(t)$ as

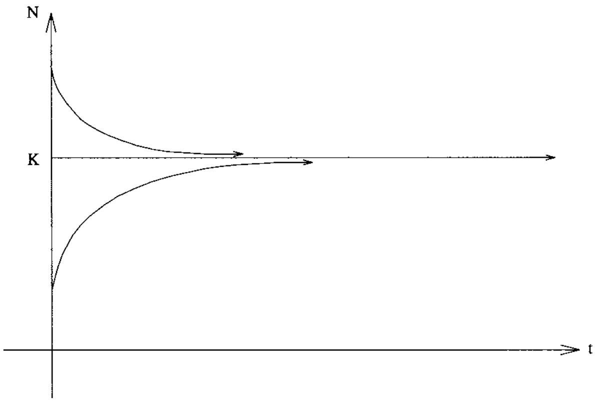

Figure 2. Population decay to carrying capacity.

$t \to \infty$, unless $N(0) = 0$. Pictorially, we have the situation qualitatively depicted in Figure 2. (The arrow indicates the direction of motion as t increases.) We naturally say that all positive initial values $N(0)$ lie in the *domain of attraction* of the stable *steady state* or equilibrium K (which is only attained as t becomes unbounded, unless $N(t) \equiv K$).

Readers are urged to investigate solutions of differential equations like (1.10) for a variety of initial conditions using readily available software such as Matlab or Mathematica to obtain and graph solutions (see, e.g., Polking (1995)). Much insight into the behavior of solutions can be gained by such experimentation, a possibility not available to those first studying these equations in the eighteenth century or to those who modeled various populations in the early twentieth century.

We can explicitly find the solution $N(t)$ of (1.10) by *separating variables*:

$$\frac{K\,dN}{(K-N)N} = r\,dt.$$

(This does not require dividing by zero, since only the constant populations $N(t)$ can ever equal 0 or K.) Using *partial fractions* (a technique that will be of recurring value below), we rewrite this equation in terms of constants A and B,

$$\frac{K}{(K-N)N} = \frac{A}{N} + \frac{B}{K-N},$$

where A and B which must satisfy

$$K = A(K - N) + BN = AK + (B - A)N.$$

Equating coefficients of both powers of N (i.e., 0 and 1) to achieve a balance for small and large N values, we ask that $K = AK$ and $B - A = 0$, yielding $B = A = 1$. Thus, the separated differential equation implies that

$$\frac{dN}{N} + \frac{dN}{K - N} = r dt.$$

If we integrate both sides, this implies

$$\ln|N| - \ln|K - N| = rt + C$$

for some constant C, so

$$\ln\left|\frac{N(t)}{K - N(t)}\right| = rt + C.$$

Exponentiating, we get $\frac{N(t)}{K-N(t)} = \widetilde{C}e^{rt}$, where the still arbitrary constant $\widetilde{C} = \pm e^C$ now becomes specified through the initial condition as $\widetilde{C} = \frac{N(0)}{K-N(0)}$. Solving for $N(t)$, we finally obtain the explicit solution

$$N(t) = \frac{KN(0)}{N(0) + (K - N(0))e^{-rt}}, \quad t \geq 0 \tag{1.12}$$

to example 4, which remains valid as t increases as long as the denominator stays nonzero. The validity of (1.12) can be directly checked by substitution in the differential equation (1.10). Note that the constant solutions are obtained for $N(0) = 0$ and for $N(0) = K$. For $0 < N(0) < K$, both terms in the denominator remain positive, but the second decreases monotonically from $K - N(0)$ to 0 as t increases; thus, $N(t) \uparrow K^-$ as $t \to \infty$. If $N(0) > K > 0$, the denominator $Ke^{-rt} + N(0)(1 - e^{-rt})$ also remains positive for all finite t, decreasing from K to $N(0)$ as t increases, so the steady-state population K is then monotonically approached from above as $t \to \infty$. Anticipating later terminology, note that the "nonlinear" population model (1.10) is somewhat more complicated to explicitly solve than the more elementary "linear" model (1.8). The saturation obtained from the nonlinear model at the finite carrying capacity K should provide a more realistic conclusion, however, than continuing exponential growth. One must check the numbers obtained against reality (before calling the exterminator).

Formula (1.12) provides the solution of the Riccati equation (1.10) as long as it exists for any set of *parameters* K, r, and $N(0)$. If we allow $r < 0$, one can readily show that $N(t) \downarrow 0$ as t increases if $0 < N(0) < K$, whereas

$N(t) \uparrow \infty$ as t increases for $N(0) > K$. The constant solution $N(t) \equiv K$ for $N(0) = K$ is naturally called *unstable* to perturbations of the initial value, because a small change in $N(0)$ away from K ultimately results in a far different solution value $N(t)$; the solution $N(t) \equiv 0$, by contrast, is a *stable* constant solution. An equivalent conclusion would be reached for $r > 0$, if we instead let t decrease from zero, but then we would be determining the population of ancestors rather than descendents, as geneologists might prefer. The important concepts of stability and instability will be the practical and significant focus of some later effort.

Generally, an nth-order scalar differential equation $F\left(t, x, \frac{dx}{dt}, \dots, \frac{d^n x}{dt^n}\right) = 0$ is called *linear* if it can be written in the very restricted form

$$a_n(t)\frac{d^n x}{dt^n} + a_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \dots + a_1(t)\frac{dx}{dt} + a_0(t)x = f(t) \qquad (1.13)$$

for coefficients $a_n, a_{n-1}, \dots, a_0$, and an f that depends only on t, with the leading coefficient $a_n(t)$ being nonzero somewhere. (Note that equation (1.13) involves, in particular, no products of x and/or its derivatives.) All other nth-order scalar differential equations are called *nonlinear* (i.e., not linear). Furthermore, a linear equation (1.13) is called *homogeneous*, and it has at least the *trivial solution* $x(t) \equiv 0$, if $f(t) \equiv 0$. Otherwise, it is called *nonhomogeneous*, and doesn't have the trivial solution. Note that arbitrary *linear combinations* $\alpha x(t) + \beta z(t)$ of any solutions $x(t)$ and $z(t)$ of such a linear homogeneous equation also satisfy the equation, whatever the constants α and β. This *superposition principle* is a cornerstone of many methods to solve all kinds of linear equations.

Analogously, a first-order differential system $g\left(t, y, \frac{dy}{dt}\right) = 0$ of n scalar equations for an n-vector $y(t)$ is called linear if it can be rewritten in the special vector-matrix form

$$B(t)\frac{dy}{dt} = C(t)y + k(t). \qquad (1.14)$$

Here,

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, k(t) = \begin{pmatrix} k_1(t) \\ \vdots \\ k_n(t) \end{pmatrix}, \quad \text{and} \quad \frac{dy}{dt} = \begin{pmatrix} \frac{dy_1}{dt} \\ \vdots \\ \frac{dy_n}{dt} \end{pmatrix}$$

are all n-vectors, while

$$B(t) = \begin{pmatrix} b_{11}(t) & \dots & b_{1n}(t) \\ \vdots & & \\ b_{n1}(t) & \dots & b_{nn}(t) \end{pmatrix}$$

and $C(t)$ are $n \times n$ matrices. We will assume that B, C and k have continuous scalar entries. At *ordinary* points, where $B(t)$ is nonsingular, we can multiply (1.14) by the inverse $B^{-1}(t)$ to get a linear differential system

$$\frac{dy}{dt} = A(t)y + g(t) \tag{1.15}$$

in standard form (with $A \equiv B^{-1}C$ and $g \equiv B^{-1}k$). Alternatively, the unknown solution components $y_i(t)$ must then successively satisfy the corresponding n scalar coupled linear differential equations

$$\frac{dy_i}{dt} = a_{i1}(t)y_1(t) + \ldots + a_{in}(t)y_n(t) + g_i(t), \quad i = 1, 2, \ldots, n.$$

The linear system (1.15) is homogeneous, and has the trivial solution $y(t) \equiv 0$, when $g(t) \equiv 0$. In the *autonomous* situation when all n^2 entries of the matrix $A(t) \equiv A$ are constant, it will eventually be convenient to define the matrix exponential e^{At} as a matrix solution Y of the constant linear homogeneous system $Y' = AY$. As before, nonlinear, nonhomogeneous, and nonautonomous, respectively, simply mean not linear, not homogeneous, and not autonomous. Linear differential systems (1.14) will be shown to be quite tractable, except near values of t where the matrix $B(t)$ becomes singular (i.e., *singular* points). This is why knowing how to solve such linear problems has been critical to advancing our knowledge of physics.

Example 5 Now consider the nonlinear second-order scalar differential equation

$$\frac{d^2r}{dt^2} = -g\frac{R^2}{r^2}, \tag{1.16}$$

which describes the motion of a *free-falling body* of unit mass toward the Earth, where $g > 0$ is the acceleration of gravity, $r \geq R > 0$ is the distance from the center of the Earth, and R is the Earth's radius. (When $r \approx R$, one usually approximates the right-hand side by $-g$ and calculus provides the quadratic solution $r(t) = -\frac{g}{2}t^2 + r'(0)t + r(0)$ of the resulting linear model.) If we (cleverly) multiply equation (1.16) by $2\frac{dr}{dt}$ (which is nonzero when the body is still falling), we get $2\frac{dr}{dt}\frac{d^2r}{dt^2} = -2gR^2\frac{1}{r^2}\frac{dr}{dt}$ or

$$\frac{d}{dt}\left(\left(\frac{dr}{dt}\right)^2 - 2gR^2\frac{1}{r}\right) = 0.$$

Integrating once, we then obtain

$$\left(\frac{dr}{dt}\right)^2 = \frac{2gR^2}{r} + C,$$

which can be interpreted as a *conservation of energy* statement, where the constant $C = (r'(0))^2 - \frac{2gR^2}{r(0)}$ is the sum of a *kinetic energy* $\left(\frac{dr}{dt}\right)^2$ and a *potential energy* $-\frac{2gR^2}{r}$ (which terms separately vary with time). Observe that C is completely determined by prescribing the initial position $r(0)$ and initial velocity $r'(0)$. Solving for

$$\frac{dr}{dt} = -\sqrt{\frac{2gR^2}{r} + C},$$

with the negative velocity selected because the fall is toward the Earth, we obtain the separable first-order equation

$$\frac{dr}{dt} = -\sqrt{(r'(0))^2 + 2gR^2\left(\frac{1}{r} - \frac{1}{r(0)}\right)}$$

for $r(t)$. The implicit (decreasing) solution $r(t)$ is thereby uniquely given by

$$t = -\int_{r(0)}^{r} \frac{ds}{\sqrt{(r'(0))^2 + 2gR^2\left(\frac{1}{s} - \frac{1}{r(0)}\right)}}, \tag{1.17}$$

since $r = r(0)$ at $t = 0$. The explicit solution $r(t)$ might be determined numerically or by inverting the graph of t as a function of r. When $r = R$, the body collides with the Earth and we must certainly question the continued appropriateness of this differential equation model.

Example 6 A simple nonlinear system of differential equations expresses the *law of mass action* in chemical kinetics. Consider a reaction described symbolically by

$$A + B \xrightarrow{k} C,$$

where $k > 0$ is a given *rate constant*. Letting u, v, and w denote the (nonnegative) concentrations of the chemicals A, B, and C, respectively, we obtain a system of three coupled nonlinear first-order differential equations

$$\begin{cases} \frac{du}{dt} = -kuv \\ \frac{dv}{dt} = -kuv \\ \frac{dw}{dt} = kuv, \end{cases} \tag{1.18}$$

assuming that the rates of change of all of the concentrations are equal to the product of the concentrations of the reactants A and B times the rate constant. If $u(0) > 0$ and $v(0) > 0$ are prescribed and $w(0) = 0$, we will have $\frac{d}{dt}(u + w)$