Introduction

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1.1 Fluids and plasmas in the astrophysical context

When a beginning student takes a brief look at an elementary textbook on fluid mechanics and at an elementary textbook on plasma physics, he or she probably forms the impression that these two subjects are very different from each other. Let us begin with some comments why we have decided to treat these two subjects together in this volume and why astrophysics students should learn about them.

We know that all substances are ultimately made up of atoms and molecules. Ordinary fluids like air or water are made up of molecules which are electrically neutral. By heating a gas to very high temperatures or by passing an electric discharge through it, we can break up a large number of molecules into positively charged ions and negatively charged electrons. Such a collection of ions and electrons is called a plasma, provided it satisfies certain conditions which we shall discuss later. Hence a plasma is nothing but a special kind of fluid in which the constituent particles are electrically charged.

When we watch a river flow, we normally do not think of interacting water molecules. Rather we perceive the river water as a continuous substance flowing smoothly as a result of the macroscopic forces acting on it. Engineers and meteorologists almost always deal with fluid flows which can be adequately studied by modelling the fluid as a continuum governed by a set of macroscopic equations. Usually most of the elementary fluid mechanics textbooks deal with these macroscopic equations without ever bothering about the molecular constitution of fluids. On the other hand, very often results of laboratory plasma experiments can be understood best in terms of forces acting on individual plasma particles and their motions. Hence elementary plasma physics textbooks often start from the dynamics of plasma particles.

Because of these very different approaches, elementary textbooks often hide the underlying unity in the sciences of fluids and plasmas.

It is intuitively obvious to us that fluids like water and air can be treated as macroscopic continuum systems. But astrophysicists often deal with systems like the solar wind or the interstellar medium having few particles per cm^3 but extending over vast regions of space. It is not at once obvious if continuum fluid equations are applicable to such systems. Hence it is useful for astrophysicists to have some understanding of the microscopic basis of the continuum equations to know when they are applicable and when they break down. We shall try to understand in this book why and under what circumstances collections of particles can be modelled as continua. Since we shall develop both the particle and continuum aspects of the theory, it is useful to approach fluids and plasmas from a unified point of view, which is often obscured in elementary textbooks by stressing the continuum aspects of neutral fluids and particle aspects of plasmas.

Most objects in the astrophysical Universe are made up of ionized material which can be regarded as plasma. Hence it is no wonder that astrophysicists have to learn about plasmas to understand how the Universe works. Often, however, the ordinary fluid dynamics equations are adequate if electromagnetic interactions are not important in a problem. We have seen that a plasma is a special kind of fluid in which the constituent particles are charged. Hence the special character of plasmas becomes apparent only in circumstances in which electromagnetic interactions play important roles. When electromagnetic interactions are unimportant, plasmas behave very much like neutral fluids which obey simpler equations. Stellar structure and oscillations are examples of important astrophysical problems for which ordinary fluid equations are *almost* adequate, even though stars are made up of plasma. If the star has a strong magnetic field, it may be necessary to apply very small plasma corrections. One of the current research topics in the study of solar oscillations is to understand the *very small* effect of magnetic fields on these oscillations.

Since neutral fluid equations in a sense can be thought to constitute a special case of plasma equations in which the electromagnetic terms are set to zero, there may be some logical appeal in first developing the full plasma equations in complete glory and then considering the neutral fluids as a special case. For pedagogical reasons, however, we have decided to present things in the opposite order. The first half of the book is devoted to neutral fluids, which obey simpler equations than plasmas. Then, in the second half, we develop the theory of plasmas, which are governed by more complicated and more general equations. Within each half, we begin from microscopic or particle considerations and then develop the continuum models. It will be seen that the microscopic theory of neutral fluids is *not* exactly of the nature of a special case of the microscopic theory of plasmas with electromagnetic forces set to zero. The particles in a neutral fluid interact only when they collide, whereas the particles in a plasma interact through long-range electromagnetic interactions. This difference in the nature of interactions introduces some subtle differences in the microscopic theories.

Although we shall be considering astrophysical applications as examples throughout the text, we want to emphasize that what we present in this book is nothing but *standard* fluid mechanics and *standard* plasma physics. Astrophysical problems often necessitate the application of the basic theory to situations very different from any terrestrial situation, but the basic physics does not change. Although the material is presented in this book in a way which would be most suitable for somebody embarking on a career of astrophysics research, a careful reader of this book should be in a position to appreciate laboratory problems in fluid mechanics and plasma physics equally well.

1.2 Characteristics of dynamical theories

We would like to develop dynamical theories of fluids and plasmas. By *dynamical theory* we mean a physical theory with which the time evolution of a system can be studied. Classical mechanics, classical electrodynamics and quantum mechanics are some of the familiar examples of dynamical theories in physics. The structures of all these dynamical theories have certain common characteristics, which we would expect the dynamical theories of fluids and plasmas also to have. Let us begin by noting down these common characteristics.

First of all, we must have a way of describing the state of our system at one instant of time. For a mechanical system, this is done by specifying all the generalized position and momentum coordinates. The state of an electromagnetic field is given by $\mathbf{E}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ at all points at an instant of time. For a quantum system, the state is prescribed by the wave function $\psi(\mathbf{x})$. In other words, the state is always prescribed by giving the numerical values of a set of variables. The second requirement for a dynamical theory is that we should have a set of equations which tells us how these variables change with time. Once such a set of equations is given, if we know the values of all the variables prescribing the state of the system at one instant of time, we shall be able to calculate the values of all these variables at some future time. In other words, it is possible to calculate some future final state of the system from the initial state. In classical mechanics, Hamilton's equations give the time derivatives of position and momentum coordinates. Maxwell's equations contain the terms $\partial \mathbf{E}/\partial t$, $\partial \mathbf{B}/\partial t$ and hence provide the dynamical theory for the electromagnetic field. For a quantum system, time-dependent Schrödinger's equation tells us how $\psi(\mathbf{x})$ changes with time.

The mathematical theories for fluids and plasmas also should have similar structures with these two characteristics:

1 There should be a way to prescribe the state of the system with a set of variables.
2 There should be a set of equations giving the time derivatives of these variables.

We may begin by asking the question how the state of a fluid or a plasma can be prescribed at an instant of time. As we have already seen, there are different levels of looking at fluids and plasmas. At a certain level, they can be regarded as collections of particles. On another level, they can be treated as continua. We expect different dynamical theories at different levels having the two general characteristics listed above. The dynamical theories at different levels should also have some correspondence amongst them. In the next section, §1.3, we give a brief outline of the different levels at which we wish to look at fluids and plasmas, and the different dynamical theories that we wish to develop at these different levels. Section 1.3 should serve as a kind of guide map for this book.

Let us end this section by commenting that these two requirements for dynamical theories can be given geometrical representations by introducing a *phase space*. A phase space is an imaginary space having many dimensions such that each of the variables necessary to prescribe the state of the system corresponds to one dimension. Since continuous functions like $\psi(\mathbf{x})$ have to be specified at all the spatial points within a certain volume (i.e. at an infinite number of points), the corresponding phase space must have infinite dimensions, each dimension corresponding to the value of ψ at one point. It is easy to see that a state of the system corresponds to one point in the phase space. Since the dynamical equations tell us how the state changes with time, they make this point in phase space move with time and trace out a trajectory.

Table 1.1 *Different levels of theory for neutral fluids and plasmas*

Neutral fluids		
Level	Description of state	Dynamical equations
0: N quantum particles	$\psi(\mathbf{x}_1, \ldots, \mathbf{x}_N)$	Schrödinger's eqn.
1: N classical particles	$(\mathbf{x}_1, \ldots, \mathbf{x}_N, \mathbf{u}_1, \ldots, \mathbf{u}_N)$	Newton's laws
2: Distribution function	$f(\mathbf{x}, \mathbf{u}, t)$	Boltzmann eqn.
3: Continuum model	$\rho(\mathbf{x}), T(\mathbf{x}), \mathbf{v}(\mathbf{x})$	Hydrodynamic eqns.

Plasmas (Levels 0 and 1 same as above)		
Level	Description of state	Dynamical equations
2: Distribution function	$f(\mathbf{x}, \mathbf{u}, t)$	Vlasov eqn.
$2\frac{1}{2}$: Two-fluid model	See Chapter 11	
3: One-fluid model	$\rho(\mathbf{x}), T(\mathbf{x}), \mathbf{v}(\mathbf{x}), \mathbf{B}(\mathbf{x})$	MHD eqns.

1.3 Different levels of theory

Since fluids and plasmas are collections of particles, let us consider a collection of N particles and look at the different levels at which one may wish to develop dynamical theories for this system. These different levels are summarized in Table 1.1. At a very fundamental level, all microscopic particles obey quantum mechanics. Let us call it Level 0. The state of the system at this level is given by the N-particle wave function, which evolves in time according to Schrödinger's equation. In this book, however, we shall not discuss this level at all. At the next higher Level 1, the system can be modelled as a collection of N classical particles. Can we always pass on from Level 0 to Level 1? No, one often encounters collections of particles which are inherently quantum and a classical description is not adequate. The electron gas within a metal is an example of such a system from everyday life and the material inside a white dwarf star is an astrophysical example. Since we are not going to discuss Level 0 in this book, the dynamics of quantum gases remains outside the scope of this book.

Under what circumstances is a description at Level 1 possible for our system of N particles? Basically the wave packets for the different particles have to be widely separated so that quantum interference is not important. If p is the typical momentum of the particles, then the de Broglie wavelength is

$$\lambda = \frac{h}{p} \approx \frac{h}{\sqrt{m\kappa_B T}},$$

where m is the mass of the particle, κ_B the Boltzmann constant and T the temperature (see, for example, Schiff 1968, p. 3; Mathews and Venkatesan 1976, §1.13). Since this is also a measure of the sizes of wave packets of individual particles, we have to compare this with the typical inter-particle distance, which is $n^{-1/3}$ if n is the particle number density per unit volume. Hence the condition for the non-overlapping of wave packets is

$$\frac{hn^{1/3}}{\sqrt{m\kappa_B T}} \ll 1. \tag{1.1}$$

When this condition is satisfied, an individual wave packet evolves according to Schrödinger's equation in an isolated fashion and can be shown to move like a classical particle. This result is known as Ehrenfest's theorem and is derived from Schrödinger's equation in any textbook on quantum mechanics (see, for example, Schiff 1968, pp. 28–30; Mathews and Venkatesan 1976, §2.7). Hence (1.1) gives the condition that Level 1 can be *derived* from Level 0. We then have at Level 1 a system of N classical particles of which the state is prescribed by the position and velocity coordinates $(\mathbf{x}_1, \ldots, \mathbf{x}_N, \mathbf{u}_1, \ldots, \mathbf{u}_N)$. The time evolution of this system can be studied by Newton's laws of motion or by Hamilton's equations.

If N is large, then it is not realistic to solve the equations of motion for all the position and velocity coordinates. Hence, in the next higher Level 2, one introduces the distribution function $f(\mathbf{x}, \mathbf{u}, t)$ giving the particle number density in the six-dimensional $(\mathbf{x}, \mathbf{u})$ space at time t ($\mathbf{x}$ is the position coordinate of a particle and $\mathbf{u}$ is its velocity coordinate). A dynamical theory at this level requires an equation which tells us how $f(\mathbf{x}, \mathbf{u}, t)$ changes in time. The time derivative of $f(\mathbf{x}, \mathbf{u}, t)$ for a neutral fluid is given by the Boltzmann equation. The corresponding equation for plasmas is called the Vlasov equation. We shall see that this equation superficially resembles the Boltzmann equation, but has some subtle differences.

At the final Level 3, we model the systems as continua. Let us first consider how the state of a neutral fluid in the continuum model can be prescribed. We know that a single-component gas in thermodynamic equilibrium can be described by two thermodynamic variables. A moving fluid is not in thermodynamic equilibrium as a whole. But if we consider a small element of fluid and go to the frame in which it is at rest, then we can regard that element to be in *approximate* thermodynamic equilibrium in that frame. This idea and the exact meaning of the adjective *approximate* will be made clearer in Chapter 3, where we derive Level 3 from Level 2. Hence the state of that element of fluid can be prescribed by two thermodynamic variables and the velocity of that element with respect to some frame, say the laboratory frame of reference. Since we have to specify the state of each and every element of the fluid in this fashion, the state of the whole fluid is given by prescribing the two thermodynamic variables and the velocity at all points of the fluid. Taking density and temperature as examples of two thermodynamic variables, the specification of $\rho(\mathbf{x})$, $T(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$ at all points of the fluid at an instant of time gives the state of the fluid at that time. The usual macroscopic hydrodynamic equations tell us how all these variables vary in time and hence constitute a complete dynamical theory for neutral fluids at Level 3.

Since plasmas can have magnetic fields embedded in them, we have to take $\mathbf{B}(\mathbf{x})$ as an additional variable when considering the Level 3 for plasmas. We know that one takes the electric field $\mathbf{E} = 0$ inside conductors when solving electrostatics problems. Since plasmas are good conductors of electricity, electric fields in the local rest frames inside plasmas are also quickly shorted by currents and it is not necessary to take the electric field as an extra variable in the continuum model at Level 3. A state of the plasma at this level can be given by prescribing $\rho(\mathbf{x})$, $T(\mathbf{x})$, $\mathbf{v}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ at all points. We shall later derive a set of equations called the *magnetohydrodynamic* or MHD equations giving the time evolutions of these variables. They are more complicated than the ordinary fluid dynamics equations. But is it always justified to ignore the electric field? It turns out that one can have electric fields in plasmas over short distances existing for short times. To handle such situations, we introduce an intermediate Level $2\frac{1}{2}$ for plasmas. At this intermediate level, we regard plasmas as mixtures of two fluids having opposite electrical charges. The details of this two-fluid model will be discussed in Chapter 11. When we consider *slow* motions of plasmas under mechanical and magnetic stresses, MHD equations are adequate. Again the exact meaning of *slow* will be made clear later. Many astrophysical problems can be handled with MHD equations. Propagation of electromagnetic waves in plasmas, however, is a problem for which it is necessary to deal with the more complex two-fluid model at Level $2\frac{1}{2}$.

We have seen that the condition (1.1) has to be satisfied in order to pass from Level 0 to Level 1. Similarly some other conditions have to be met to derive Level 2 from Level 1 or Level 3 from Level 2. These conditions will be discussed in the appropriate places of the book. If a system of N particles satisfies these conditions, then it is possible to introduce the distribution function $f(\mathbf{x}, \mathbf{u}, t)$ or to model the system as a continuum.

Much of this book is devoted to studying the dynamics of neutral fluids and plasmas at Levels 2 and 3 (with the additional Level $2\frac{1}{2}$ for plasmas). To begin with, however, we need to understand how we can develop Level 2 from Level 1. For a proper appreciation of this subject, it is important to know some general results pertaining to phase spaces of dynamical systems. In view of the generality of these results, we have decided to discuss them in the next two sections of this introductory chapter and end the chapter with them.

We now end this section with a comment on predictability. It would seem that a dynamical theory satisfying the structural requirements described in §1.2 would be completely predictable. In other words, knowing the present state of the system, one would always be able to predict the future completely. Fluids and plasmas, however, can often have *turbulence*—a state of random and chaotic motions which appear unpredictable. Developing a proper theory of turbulence has remained one of the unsolved grand problems of physics for over a century. We shall discuss in Chapter 8 the question of how turbulence can arise in systems apparently governed by predictable equations. Even if a dynamical theory is predictable *in principle*, we shall see that there can be a loss of predictability *in practice*.

1.4 Ensembles in phase space. Liouville's theorem

Let us consider a dynamical system of which a state can be prescribed by the generalized position and momentum coordinates $(q_s, p_s; s = 1, \ldots, n)$ and which evolves according to Hamilton's equations:

$$\dot{p}_s = -\frac{\partial H}{\partial q_s}, \tag{1.2}$$

$$\dot{q}_s = \frac{\partial H}{\partial p_s}, \tag{1.3}$$

where the Hamiltonian $H(q_s, p_s, t)$ can be a function of all the coordinates and time (see, for example, Goldstein 1980, Chapter 8; Raychaudhuri 1983, Chapters 8–9). If you do not have a deep understanding of Hamiltonian theory, you need not panic. We shall make use of Hamiltonian theory only rarely in this book and an acquaintance with the above two equations will suffice.

Considering Hamiltonian systems may seem somewhat restrictive, because not all dynamical theories can be put in the Hamiltonian form.

Readers familiar with the subject would know that it is not possible to make a Hamiltonian formulation of a dissipative system. However, dissipation in macroscopic systems usually means that the energy of some ordered macroscopic motion is being transferred into random molecular motions. When we look at a system at the microscopic level (say our Level 1) and include the molecular motions within the fold of the dynamical theory, usually a Hamiltonian formulation is possible. Our system at Level 1, a collection of N classical particles, certainly allows a Hamiltonian treatment.

For the statistical treatment of a system, it is often useful to introduce the concept of an *ensemble*. An ensemble means a set of many replicas of the same system, which are identical in all other respects apart from being in different states at an instant of time. Hence each member of the ensemble can be represented by a point in the phase space at an instant of time and their evolutions correspond to different trajectories in the phase space. If the ensemble points are distributed sufficiently densely and smoothly in the phase space, then it is meaningful to talk about the density of ensemble points at a location in the phase space. Let us denote this density by $\rho_{\text{ens}}(q_s, p_s, t)$.

We now wish to prove Liouville's theorem, which is one of the fundamental theorems of statistical mechanics. Let us first state the theorem. Then we shall proceed to prove it. Let us consider one member of the ensemble and its trajectory $(q_s(t), p_s(t))$ in the phase space. We keep measuring the density $\rho_{\text{ens}}(q_s(t), p_s(t), t)$ as a function of time varying as a parameter along this trajectory. Liouville's theorem states that the time derivative of this density as we move along the trajectory is zero, i.e.

$$\frac{D\rho_{\text{ens}}}{Dt} = 0, \tag{1.4}$$

where D/Dt denotes the time derivative along the trajectory. If (q_s, p_s) and $(q_s + \delta q_s, p_s + \delta p_s)$ denote the states of the system at times t and $t + \delta t$ on this trajectory, then

$$\frac{D\rho_{\text{ens}}}{Dt} = \lim_{\delta t \to 0} \frac{\rho_{\text{ens}}(q_s + \delta q_s, p_s + \delta p_s, t + \delta t) - \rho_{\text{ens}}(q_s, p_s, t)}{\delta t}. \tag{1.5}$$

Expansion in a Taylor series to linear terms in small quantities gives

$$\rho_{\text{ens}}(q_s + \delta q_s, p_s + \delta p_s, t + \delta t) = \rho_{\text{ens}}(q_s, p_s, t) + \sum_s \delta q_s \frac{\partial \rho_{\text{ens}}}{\partial q_s} + \sum_s \delta p_s \frac{\partial \rho_{\text{ens}}}{\partial p_s} + \delta t \frac{\partial \rho_{\text{ens}}}{\partial t}.$$