CHAPTER I

THE TYPE PROBLEM

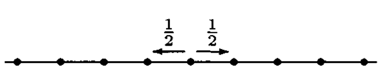
1. Basic facts

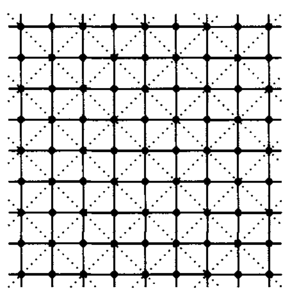
Before embarking on a review of the basic material concerning Markov chains, graphs, groups, etc., let us warm up by considering the classical example.

A. Pólya's walk

The d-dimensional grid, denoted briefly by $\mathbb{Z}^d$, is the graph whose vertices are integer points in d dimensions, and where two points are linked by an edge if they are at distance 1. A walker wanders randomly from point to point; at each "crossroad" (point) he chooses with equal probability the one among the $2d$ neighbouring points where his next step will take him, see Figure 1. Starting from the origin, what is the probability $p^{(2n)}(0,0)$ that the walker will be back at the $2n$th step? This is the number of closed paths of length $2n$ starting at the origin, divided by $(2d)^{2n}$. (The walker cannot be back after an odd number of steps.) For small dimensions, the solutions of this combinatorial exercise are as follows.

Figure 1: the grids $\mathbb{Z}$ and $\mathbb{Z}^2$





$d = 1$. Among the $2n$ steps, the walker has to make n to the left and n to the right. Hence

$$p^{(2n)}(0,0) = \frac{1}{2^{2n}}\binom{2n}{n} \sim C_1\, n^{-1/2}\,. \tag{1.1}$$

$d = 2$. Let two walkers perform the one-dimensional random walk simultaneously and independently. Their joint trajectory, viewed in $\mathbb{Z}^2$, visits only the set of points (i, j) with $i + j$ even. However, the graph with this set of vertices, and with two points neighbours if they differ by ± 1 in each component, is isomorphic with the grid $\mathbb{Z}^2$ and probabilities are preserved under this isomorphism. Hence

$$(1.2) \qquad p^{(2n)}(0,0) = \left(\frac{1}{2^{2n}}\binom{2n}{n}\right)^2 \sim C_2\, n^{-1}\,.$$

$d = 3$. It is no longer possible to represent the random walk in terms of three independent random walks on $\mathbb{Z}$. In a path of length $2n$ starting and ending at the origin, n steps have to go north, east, or up. There are $\binom{2n}{n}$ possibilities to assign the n steps of these three types; the other n go south, west, or down. For each of these choices, i steps go north and i go south, j steps go east and j go west, $n-i-j$ steps go up and $n-i-j$ go down. Hence

$$p^{(2n)}(0,0) = \frac{1}{6^{2n}}\binom{2n}{n}\sum_{i+j\le n}\left(\frac{n!}{i!\,j!\,(n-i-j)!}\right)^2\,.$$

Consider the function $(x,y,z) \mapsto x!y!z!$ for $x,y,z \ge 0$. Under the condition $x+y+z=n$, it assumes its minimum for $x=y=z=n/3$, when n is sufficiently large. Hence

$$(1.3) \qquad \begin{aligned} p^{(2n)}(0,0) &\le \frac{1}{6^{2n}}\binom{2n}{n}\frac{n!}{(n/3)!^3}\sum_{i+j\le n}\left(\frac{n!}{i!\,j!\,(n-i-j)!}\right) \\ &= \frac{1}{6^{2n}}\binom{2n}{n}\frac{n!}{(n/3)!^3}\,3^n \sim C_3\, n^{-3/2}\,. \end{aligned}$$

Indeed, for arbitrary dimension d, there are various ways to show that

$$(1.4) \qquad p^{(2n)}(0,0) \sim C_d\, n^{-d/2}\,.$$

Now for the random walk starting at the origin, $\sum_n p^{(2n)}(0,0)$ is the expected number of visits of the walker back to the origin: this is infinite for $d=1,2$ and finite for $d \ge 3$. This drastic change of behaviour from two to three dimensions stands at the origin of our investigations.

B. Irreducible Markov chains

A *Markov chain* is (in principle) given by a finite or countable *state space* X and a stochastic *transition matrix* (or *transition operator*) $P = \big(p(x,y)\big)_{x,y\in X}$. In addition, one has to specify the starting point (or a starting distribution on X). The matrix element $p(x,y)$ is the probability of moving from x to y in one step. Thus, we have a sequence of X-valued random variables Z_n, $n \ge 0$, with Z_n representing the random position in X at time n. To model Z_n, the usual choice of probability space is the *trajectory space* $\Omega = X^{\mathbb{N}_0}$, equipped with the product σ-algebra arising from the discrete one on X. Then Z_n is the nth projection $\Omega \to X$. This describes the Markov chain starting at x, when Ω is equipped with the probability measure given via the Kolmogorov extension theorem by

$$\mathbb{P}_x[Z_0 = x_0, Z_1 = x_1, \ldots, Z_n = x_n] = \delta_x(x_0)p(x_0, x_1)\cdots p(x_{n-1}, x_n).$$

The associated expectation is denoted by $\mathbb{E}_x$. Alternatively, we shall call a Markov chain (random walk) the pair (X, P) or the sequence of random variables $(Z_n)_{n\geq 0}$. We write

$$p^{(n)}(x,y) = \mathbb{P}_x[Z_n = y].$$

This is the (x,y)-entry of the matrix power P^n, with $P^0 = I$, the identity matrix over X. Throughout this book, we shall always require that all states communicate:

(1.5) Basic assumption. (X, P) *is irreducible, that is, for every* $x, y \in X$ *there is some* $n \in \mathbb{N}$ *such that* $p^{(n)}(x,y) > 0$.

Next, we define the *Green function* as the power series

$$G(x,y|z) = \sum_{n=0}^{\infty} p^{(n)}(x,y)\, z^n, \quad x, y \in X, z \in \mathbb{C}. \tag{1.6}$$

(1.7) Lemma. *For real* $z > 0$, *the series* $G(x,y|z)$ *either diverge or converge simultaneously for all* $x, y \in X$.

Proof. Given $x_1, y_1, x_2, y_2 \in X$, by irreducibility there are $k, \ell \in \mathbb{N}$ such that $p^{(k)}(x_1, x_2) > 0$ and $p^{(\ell)}(y_2, y_1) > 0$. We have

$$p^{(k+n+\ell)}(x_1, y_1) \geq p^{(k)}(x_1, x_2)p^{(n)}(x_2, y_2)p^{(\ell)}(y_2, y_1)$$

and hence, for $z > 0$,

$$G(x_1, y_1|z) \geq p^{(k)}(x_1, x_2)p^{(\ell)}(y_2, y_1)z^{k+\ell}G(x_2, y_2|z). \qquad \square$$

As a consequence, all the $G(x,y|z)$ (where $x, y \in X$) have the same radius of convergence $\mathrm{r}(P) = 1/\rho(P)$, given by

$$\rho(P) = \limsup_{n\to\infty} p^{(n)}(x,y)^{1/n} \in (0\,,\,1]. \tag{1.8}$$

This number is often called the *spectral radius* of P.

The *period* of P is the number $\mathsf{d} = \mathsf{d}(P) = \gcd\{n \geq 1 : p^{(n)}(x,x) > 0\}$. It is well known and easy to check that it is independent of x by irreducibility. If $\mathsf{d}(P) = 1$ then the chain is called *aperiodic*. Choose $o \in X$ and define $Y_j = \{x \in X : p^{(n\mathsf{d}+j)}(o,x) > 0 \text{ for some } n \geq 0\}$, $j = 0, \ldots, \mathsf{d}-1$. This defines a partition of X, and x, y are in the same class if and only if $p^{(n\mathsf{d})}(x,y) > 0$ for some n. These are the periodicity classes of (X, P), visited by the chain $(Z_n)_{n\geq 0}$ in cyclical order. The restriction of P^{d} to each class is irreducible and aperiodic. See e.g. Chung [75] for these facts.

(1.9) Lemma. $p^{(n)}(x,x) \le \rho(P)^n$, *and* $\lim_{n\to\infty} p^{(n\mathrm{d})}(x,x)^{1/n\mathrm{d}} = \rho(P)$.

Proof. Write $a_n = p^{(n\mathrm{d})}(x,x)$. Then $0 \le a_n \le 1$ and $\gcd N(x) = 1$, where $N(x) = \{n : a_n > 0\}$. The crucial property is $a_m a_n \le a_{m+n}$.

We first show that there is n_0 such that $a_n > 0$ for all $n \ge n_0$. If $m, n \in N(x)$ then $m+n \in N(x)$. Recall that the greatest common divisor of a set of integers can always be written as a finite linear combination with integer coefficients of elements of that set. Therefore we can write $1 = \gcd N(x) = n_1 - n_2$, where $n_1, n_2 \in N(x) \cup \{0\}$. If $n_2 = 0$ we are done ($n_0 = 1$). Otherwise, set $n_0 = n_2^2$ and decompose $n \ge n_0$ as $n = q\,n_2 + r = (q-r)n_2 + r\,n_1$, where $0 \le r < n_2$. It must be that $q \ge n_2 > r$, so that $n \in N(x)$. Next, fix $m \in N(x)$, let $n \ge n_0 + m$, and decompose $n = q_n m + r_n$, where $n_0 \le r_n < n_0 + m$. Write $b = b(m) = \min\{a_r : n_0 \le r < n_0 + m\}$. Then $b > 0$ and $a_n \ge a_m^{q_n} a_{r_n}$, so that $a_m^{q_n/n} b^{1/n} \le a_n^{1/n}$. If $n \to \infty$ then $q_n/n \to m$. Hence,

$$a_m^{1/m} \le \liminf_{n\to\infty} a_n^{1/n} \le \rho(P)^{\mathrm{d}} \quad \text{for every } m \in N(x)\,.$$

This proves the first statement. If we now let $m \to \infty$, then $\limsup_m a_m^{1/m} \le \liminf_n a_n^{1/n}$, and $a_n^{1/n}$ converges. □

(1.10) Exercise. Prove the following. *If P is irreducible and aperiodic then P^k is irreducible and aperiodic for every $k \ge 1$, and $\rho(P^k) = \rho(P)^k$.*

Next, define the stopping time $\mathbf{s}^y = \min\{n \ge 0 : Z_n = y\}$ (where the minimum is ∞ when the set is empty) and the hitting probabilities plus associated generating functions

$$(1.11) \qquad f^{(n)}(x,y) = \mathbb{P}_x[\mathbf{s}^y = n] \quad \text{and} \quad F(x,y|z) = \sum_{n=0}^{\infty} f^{(n)}(x,y)\, z^n\,,$$

where $z \in \mathbb{C}$. Note that $F(x,x|z) = 1$. Finally, let

$$(1.12) \qquad \mathbf{t}^x = \min\{n \ge 1 : Z_n = x\} \quad \text{and} \quad U(x,x|z) = \sum_{n=0}^{\infty} \mathbb{P}_x[\mathbf{t}^x = n]\, z^n\,.$$

The following will be useful on several occasions.

(1.13) Lemma. (a) $G(x,x|z) = \dfrac{1}{1 - U(x,x|z)}$,

(b) $G(x,y|z) = F(x,y|z) G(y,y|z)$,

(c) $U(x,x|z) = \sum_y p(x,y) z\, F(y,x|z)$ *and,*

(d) *if* $y \ne x$, $F(x,y|z) = \sum_w p(x,w) z\, F(w,y|z)$.

Proof. Part (a) follows from the identity

$$p^{(n)}(x,x) = \sum_{k=0}^{n} \mathbb{P}_x[\mathbf{t}^x = k]\, p^{(n-k)}(x,x)\,, \quad \text{if } n \geq 1\,,$$

while $p^{(0)}(x,x) = 1$ and $\mathbb{P}_x[\mathbf{t}^x = 0] = 0$.

Analogously, (b) is obtained by conditioning with respect to the first visit to y. Parts (c) and (d) are obtained by factoring though the first step (that is, the values of Z_1). □

We shall write $G(x,y)$ for $G(x,y|1)$. This is the expected number of visits of $(Z_n)_{n\geq 0}$ to y when $Z_0 = x$. Analogously, $F(x,y)$ stands for $F(x,y|1)$, the probability of ever reaching y when starting at x, and $U(x,x) = U(x,x|1) = \mathbb{P}_x[\mathbf{t}^x < \infty]$ is the probability of ever returning after starting at x.

(1.14) Definition. The Markov chain (X,P) is called *recurrent* if $G(x,y) = \infty$ for some ($\Longleftrightarrow$ every) $x,y \in X$, or equivalently, if $U(x,x) = 1$ for some ($\Longleftrightarrow$ every) $x \in X$. Otherwise, the Markov chain is called *transient.*

If $\rho(P) < 1$ then (X,P) is transient. The converse is not true. The spectral radius will be studied in Chapter II, with sufficient transience criteria as by-products. There is a useful characterization of recurrence is terms of superharmonic functions. P acts on functions $f : X \to \mathbb{R}$ by

$$Pf(x) = \sum_y p(x,y)\, f(y)\,.$$

(We assume that $P|f|$ is finite.) We say that f is *superharmonic* if $Pf \leq f$ pointwise, and *harmonic* if $Pf = f$.

(1.15) Minimum principle. *If f is superharmonic and there is $x \in X$ such that $f(x) = \min_X f$ then f is constant.*

Proof. For every n, we have $f(x) \geq \sum_y p^{(n)}(x,y)f(y)$. Hence, it cannot be that $f(y) > f(x)$ for any y such that $p^{(n)}(x,y) > 0$. Now irreducibility yields $f \equiv f(x)$. □

For harmonic functions there is an analogous *maximum principle* (the minimum principle applied to $-f$).

(1.16) Theorem. *(X,P) is recurrent if and only if all non-negative superharmonic functions are constant.*

Proof. If (X,P) is transient then for $y \in X$, the function $x \mapsto G(x,y)$ is superharmonic, non-harmonic and hence non-constant.

Conversely, assume that (X,P) is recurrent. Let $f \geq 0$ be any superharmonic function. Set $g = f - Pf$. We claim that $g \equiv 0$. Suppose $g(y) > 0$ for some y. Choose $x \in X$. For each n,

$$\sum_{k=0}^{n} p^{(n)}(x,y)g(y) \leq \sum_{k=0}^{n} P^n g(x) = f(x) - P^{n+1}f(x) \leq f(x).$$

Consequently, $G(x,y) \leq f(x)/g(y)$ in contradiction with recurrence. We have shown that every non-negative superharmonic function is harmonic.

Now, for superharmonic $f \geq 0$, choose $x \in X$ and set $M = f(x)$. Then $h = f \wedge M$ (pointwise minimum) is superharmonic and hence also harmonic. It assumes its maximum M, and by the maximum principle, h is constant. Thus f is constant. □

Here are further characterizations of recurrence and transience.

(1.17) Proposition. (a) *If (X,P) is recurrent then $F(x,y) = 1$ and*

$$\mathbb{P}_x[Z_n = y \text{ for infinitely many } n] = 1 \quad \textit{for all } x,y \in X.$$

(b) *If (X,P) is transient then for every finite $A \subset X$,*

$$\mathbb{P}_x[Z_n \in A \text{ for infinitely many } n] = 0 \quad \textit{for all } x \in X.$$

Proof. First, observe that for $y \in X$, the function $x \mapsto F(x,y)$ is superharmonic (Lemma 1.13). Thus, in the recurrent case, $F(\cdot,y)$ is constant by Theorem 1.16, and equal to $F(y,y) = 1$.

Next, write $V(x,y) = \mathbb{P}_x[Z_n = y$ for infinitely many $n]$. Conditioning with respect to $\mathbf{s}^y$, one sees that $V(x,y) = F(x,y)V(y,y) \leq V(y,y)$. Factoring through the first step, one sees that $x \mapsto V(x,y)$ is harmonic. By the maximum principle, $V(x,y) = V(y,y)$ for all x,y.

Set $V_m(x,x) = \mathbb{P}_x[(Z_n)_{n\geq 0}$ visits x at least m times$]$. Then $V_1(x,x) = 1$, and conditioning with respect to $\mathbf{t}^x$, one sees that $V_m(x,x) = U(x,x)V_{m-1}(x,x)$. Hence

$$V(x,x) = \lim_{n\to\infty} V_m(x,x) = \lim_{n\to\infty} U(x,x)^{m-1}$$

is equal to 1 in the recurrent case and 0 in the transient case. This proves (a). Furthermore, as A is finite,

$$\mathbb{P}_x[Z_n \in A \text{ for infinitely many } n] \leq \sum_{y\in A} V(x,y),$$

which is 0 in the transient case. □

In particular, an irreducible Markov chain on a finite state space is always recurrent. We shall be interested in the case when X is infinite.

A recurrent Markov chain (X, P) is called *positive recurrent* if $\mathbb{E}_x[\mathbf{t}^x] < \infty$, and *null recurrent* if $\mathbb{E}_x[\mathbf{t}^x] = \infty$. Noting that $\mathbb{E}_x[\mathbf{t}^x] = U'(x, x|1-)$ (derivative with respect to z), it is easy to prove (similarly to Lemma 1.7) that this does not depend on the choice of $x \in X$. Before stating a criterion, we need another definition. P acts on non-negative measures on X by

$$\nu\, P(y) = \sum_x \nu(x)\, p(x, y)\,.$$

(We assume that νP is finite.) We say that ν is *excessive* if $\nu P \le \nu$ pointwise, and *invariant* if $\nu P = \nu$. (Irreducibility implies $\nu(x) > 0$ for all x if this holds for some x.) We omit the proof of the following criterion.

(1.18) Theorem. (a) *(X, P) is recurrent if and only if there is an invariant measure ν such that every positive excessive measure is a multiple of ν.*

(b) *(X, P) is positive recurrent if and only if ν has finite mass.*

The recurrent Markov chains that we shall encounter in this book will usually be null recurrent.

C. Random walks on graphs

We think of a *graph* as a finite or countable set of vertices (points) X, equipped with a symmetric *neighbourhood* or *adjacency relation* $\sim$ (a subset of $X \times X$). To view X, we draw a segment (edge), sometimes denoted by $[x, y]$, between every pair of neighbours x, y (so that $[x, y] = [y, x]$). Note that we do not exclude loops. We shall also write E or $E(X)$ for the edge set. A (finite) *path* from x to y in X is a sequence $\pi = [x = x_0, x_1, \ldots, x_k = y]$ such that $x_{i-1} \sim x_i$; the number $k \ge 0$ is its length. (Alternatively, we shall think of π as a sequence of edges.) We shall always assume that our graphs are *connected,* that is, every pair of vertices is joined by a path. Thus, X carries an integer-valued *metric:* $d(x, y)$ is the minimum among the lengths of all paths from x to y. A path from x to y is called *simple* if it has no repeated vertex, and *geodesic* if its length is $d(x, y)$. We denote by $\Pi(x, y)$ the set of all geodesics from x to y.

The *degree* $\deg(x)$ of a vertex x is its number of neighbours. With a few exceptions, we shall usually consider only graphs which are *locally finite,* that is, every vertex has finite degree. We say that X has *bounded geometry,* if it is connected with bounded vertex degrees, and that X is $(M\text{-})$*regular,* if $\deg(\cdot) \equiv M$ is constant.

The *simple random walk* on a locally finite graph X is the Markov chain with state space X and transition probabilities

$$(1.19)\qquad p(x,y) = \begin{cases} 1/\deg(x)\,, & \text{if } y \sim x\,,\\ 0\,, & \text{otherwise.} \end{cases}$$

The graph X is said to be recurrent (transient) if the simple random walk has this property. The simple random walk is the basic example of a random walk (Markov chain) adapted to the underlying structure. In the sequel, we shall consider various more general types of adaptedness properties of the transition matrix P of a Markov chain to the structure of the underlying graph X, and it is in the presence of such adaptedness properties that we speak of a random walk (instead of a Markov chain). Here is a list of some of these properties, which will be frequently used.

We say that P is of *nearest neighbour* type, if $p(x,y) > 0$ occurs only when $d(x,y) \le 1$.

The random walk is called *uniformly irreducible* if there are $\varepsilon_0 > 0$ and $K < \infty$ such that

$$(1.20)\qquad x \sim y \quad \text{implies} \quad p^{(k)}(x,y) \ge \varepsilon_0 \quad \text{for some } k \le K\,.$$

Note that this implies that $\deg(x) \le (K+1)/\varepsilon_0$ for every $x \in X$. Indeed,

$$K + 1 = \sum_{y\in X}\sum_{k=0}^{K} p^{(k)}(x,y) \ge \deg(x)\,\varepsilon_0\,.$$

When $\{y : p(x,y) > 0\}$ is finite for every x, we say that P has *finite range*. In itself, finite range is not an adaptedness property. However, this is the case for *bounded range*, that is, when

$$(1.21)\qquad \sup\{d(x,y) : x,y \in X\,,\ p(x,y) > 0\} < \infty\,.$$

This can be generalized by imposing conditions like tightness, uniform integrability, etc., on the family of *step length distributions* on $\mathbb{N}_0$. The latter are given for each $x \in X$ by

$$(1.22)\qquad \sigma_x(n) = \mathbb{P}_x[d(Z_1,Z_0) = n] = \sum_{y:d(y,x)=n} p(x,y)\,.$$

Consider the kth moment $M_k(\sigma_x) = \sum_n n^k \sigma_x(n) = \mathbb{E}_x\big(d(Z_1,Z_0)^k\big)$. We say that P has *finite kth moment*, if $M_k(P) = \sup_X M_k(\sigma_x)$ is finite, and that P has *exponential moment of order* $c > 0$, if $\sup_X \sum_n e^{cn}\sigma_x(n) < \infty$.

Further adaptedness conditions of geometric type will be introduced later on.

D. Trees

The nearest neighbour random walk on trees, and in particular the simple random walk on homogenous trees, is the other basic example besides Pólya's walk. A *tree* is a connected graph T without loops or cycles, where by a *cycle* in a graph we mean a sequence of vertices $x_0 \sim x_1 \sim \cdots \sim x_n$, $n \geq 3$, with no repetitions besides $x_n = x_0$. One characteristic feature of a tree is that for every pair of vertices x, y there is a unique path (*geodesic arc*) $\pi(x,y)$ of length $d(x,y)$ connecting the two.

Let P be the transition matrix of an irreducible nearest neighbour random walk on T. The following is a fundamental property linking tree structure and random walk.

(1.23) Lemma. *If $w \in \pi(x,y)$ then $F(x,y|z) = F(x,w|z)F(w,y|z)$.*

Proof. By the tree structure, the random walk must pass through w on the way from x to y. Conditioning with respect to the first visit in w, this yields

$$f^{(n)}(x,y) = \sum_{k=0}^{n} f^{(k)}(x,w) f^{(n-k)}(w,y). \qquad \square$$

As another "warm up" exercise, let us now consider a particularly typical example. The *homogeneous tree* $\mathbb{T}_M$ is the tree where all vertices have degree M. ($\mathbb{T}_2$ is isomorphic with $\mathbb{Z}$. See Figure 2 for $\mathbb{T}_3$.)

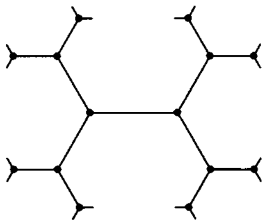


Figure 2: the homogeneous tree $\mathbb{T}_3$

(1.24) Lemma. *For the simple random walk on $\mathbb{T}_M$, one has*

$$G(x,y|z) = \frac{2(M-1)}{M-2+\sqrt{M^2-4(M-1)z^2}} \left(\frac{M-\sqrt{M^2-4(M-1)z^2}}{2(M-1)z} \right)^{d(x,y)}.$$

In particular, $\rho(P) = \dfrac{2\sqrt{M-1}}{M}$.

Proof. Obviously $F(x,y|z) = F(z)$ is the same for every pair of neighbours x, y, so that Lemma 1.23 yields $F(v,w|z) = F(z)^{d(v,w)}$. Now consider two neighbours x, y. Applying Lemma 1.13(d) we get

$$F(z) = F(x,y|z) = \sum_{w \sim x} \frac{1}{M} z\, F(z)^{d(y,w)} = \frac{1}{M} z + \frac{M-1}{M} z\, F(z)^2 .$$

This second order equation has two solutions. As $F(0) = 0$, the right one is (by continuity)

$$F(z) = \frac{1}{2(M-1)z} \left(M - \sqrt{M^2 - 4(M-1)z^2} \right) .$$

Using Lemma 1.13(c), (a) and (b), one now computes $U(x,x|z) = zF(z)$, $G(x,x|z)$ and the formula for $G(x,y|z)$.

The way in which $\rho(P)$ is read from this formula is typical: $G(x,x|z)$ is a power series with non-negative coefficients. By Pringsheim's theorem (see Hille [173], p. 133), the radius of convergence $\mathbf{r}(P) = 1/\rho(P)$ must be its smallest positive singularity. Thus, we have to compute the value of $z > 0$ where the term under the square root is equal to 0. □

As a consequence, the simple random walk on $\mathbb{T}_M$ is transient for every $M \geq 3$.

(1.25) Exercise. Compute $G(x,y|z)$ for the simple random walk on the bi-regular tree, that is, the tree where the vertex degrees are constant on each of the two bipartite classes. (These are the points at even or odd distance, respectively, from a given reference vertex.)

E. Random walks on finitely generated groups

Pólya's walk, besides being the simple random walk on a graph (the d-dimensional grid), can also be interpreted in terms of groups. The same is true for the simple random walk on $\mathbb{T}_M$.

Let Γ be a discrete group with unit element o (the symbol e will be used for edges), and let μ be a probability measure on Γ. The *(right) random walk* on Γ with *law* μ is the Markov chain with state space Γ and transition probabilities

$$p(x,y) = \mu(x^{-1}y) .$$

(Unless Γ is abelian, the group operation will be written multiplicatively.) Besides the trajectory space, in this case we may also use the product space $(\Gamma, \mu)^{\mathbb{N}}$ to obtain an equivalent model of (Z_n): the nth projections X_n of $\Gamma^{\mathbb{N}}$ onto Γ $(n \geq 1)$ constitute a sequence of independent Γ-valued random