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To my parents — Grace E. Olver and Frank W.J. Olver

As all roads lead to Rome so I find in my own case at least that all algebraic inquiries, sooner or later, end at the Capitol of Modern Algebra over whose shining portal is inscribed the Theory of Invariants.

— Sylvester, quoted in [**72**; p. 143].

The *theory of invariants* came into existence about the middle of the nineteenth century somewhat like Minerva: a grown-up virgin, mailed in the shining armor of algebra, she sprang forth from Cayley's Jovian head. Her Athens over which she ruled and which she served as a tutelary and beneficent goddess was *projective geometry*.

— Weyl, [**230**].

Like the Arabian phoenix arising out of its ashes, the theory of invariants, pronounced dead at the turn of the century, is once again at the forefront of mathematics.

— Kung and Rota, [**135**].

Contents

Introduction

Classical invariant theory is the study of the intrinsic or geometrical properties of polynomials. This fascinating and fertile field was brought to life at the beginning of the last century just as the theory of solubility of polynomials was reaching its historical climax. It attained its zenith during the heyday of nineteenth-century mathematics, uniting researchers from many countries in a common purpose, and filling the pages of the foremost mathematical journals of the time. The dramatic and unexpected solution to its most fundamental problem — the finitude of the number of fundamental invariants — propelled the young David Hilbert into the position of the most renowned mathematician of his time. Following a subsequent decline, as more fashionable subjects appeared on the scene, invariant theory sank into obscurity during the middle part of this century, as the abstract approach entirely displaced the computational in pure mathematics. Ironically, though, its indirect influence continued to be felt in group theory and representation theory, while in abstract algebra the three most famous of Hilbert's general theorems — the Basis Theorem, the Syzygy Theorem, and the Nullstellensatz — were all born as lemmas (*Hilfsätze*) for proving "more important" results in invariant theory! Recent years have witnessed a dramatic resurgence of this venerable subject, with dramatic new applications, ranging from topology and geometry, to physics, continuum mechanics, and computer vision. This has served to motivate the dusting off of the old computational texts, while the rise of computer algebra systems has brought previously infeasible computations within our grasp. In short, classical invariant theory is the closest we come in mathematics to sweeping historical drama and romance. As a result, the subject should hold a particular fascination, not only for the student and practitioner, but also for any mathematician with a desire to understand the culture, sociology, and history of mathematics.

I wrote this introductory textbook in the hope of furthering the recent revival of classical invariant theory in both pure and applied mathematics. The presentation is not from an abstract, algebraic standpoint, but rather as a subject of interest for applications in both mathematics and other scientific fields. My own training is in differential equations and mathematical physics, and so I am unashamed to restrict my attention to just real and complex polynomials. This approach allows me to directly employ differentiation and other analytical tools as the need arises. In this manner, the exposition at times resembles that of the classical texts from the last century, rather than that of more modern treatments that either presuppose an extensive training in the methods of abstract algebra or reduce the subject to a particular case of general tensor analysis. Nevertheless, a fair amount of more recent material and modern developments is covered, including several original results that have not appeared in print before. I have designed the text so that it can be profitably read by students having a fairly minimal number of mathematical prerequisites.

Notes to the Reader

The purpose of this book is to provide the student with a firm grounding in the basics of classical invariant theory. The text is written in a non-abstract manner and makes fairly low demands on the prospective reader. In addition, a number of innovations — in methodology, style, and actual results — have been included that should attract the attention of even the most well-seasoned researcher. We shall concentrate on the basic theory of binary forms, meaning polynomials in a single variable, under the action of the projective group of linear fractional transformations, although many of the methods and theoretical foundations to be discussed have far wider applicability. The classical constructions are all founded on the theory of groups and their representations, which are developed in detail from the beginning during the exposition.

The text begins with the easiest topic of all: the theory of a single real or complex quadratic polynomial in a single variable. Although completely elementary, this example encapsulates the entire subject and is well worth reviewing one more time — although an impatient reader can entirely omit this preliminary chapter. As any high school student knows, the solution to the quadratic equation relies on the associated discriminant. Less obvious is the fact that the discriminant is (relatively) unchanged under linear fractional transformations. Hence it forms the first (both historical and mathematical) example of an *invariant* and so can be used for classification of canonical forms. The text starts in earnest in Chapter 2, which provides an overview of the basics of classical invariant theory within the context of binary forms. Here we meet up with the basic definitions of invariants and covariants, and investigate how the geometry of projective space governs the correspondence between homogeneous and inhomogeneous polynomials, as well as their transformation properties under, respectively, linear and projective transformations. The motivating examples of cubic and quartic polynomials are discussed in detail, including complete lists of invariants, covariants, and canonical forms. The Fundamental Theorem of Algebra guarantees the existence of a complete system of (complex) roots, whose geometrical configuration is governed by the invariants. Two particularly important invariants are the classical resultant, which indicates the existence of common roots to a pair of polynomials, and the discriminant, which indicates multiple roots of a single polynomial. The chapter concludes with a brief introduction to the Hilbert Basis Theorem, which states that every system of polynomials has only a finite number of polynomially independent invariants, along with remarks on the classification of algebraic relations or syzygies among the invariants.

With this preliminary survey as our motivating guide, the next two chapters provide a grounding in the modern mathematical foundations of the subject, namely, transformation groups and representation theory. Chapter 3 is a self-contained introduction to groups and their actions on spaces. Groups originally arose as the symmetries of a geometric or algebraic object; in our case the object is typically a polynomial. The chapter includes a discussion of the equivalence problem — when can two objects be transformed into each other by a suitable group element — and the allied concept of a canonical form. Chapter 4 concentrates on the theory of linear group actions, known as representations. For general transformation groups, the associated multiplier representations act on the functions defined on the space; the linear/projective actions on polynomials form a very particular instance of this general construction. The invariant functions arise as fixed points for such representations, and so the focus of classical invariant theory naturally falls within this general framework.

The next three chapters describe the core of the classical constructive algebraic theory of binary forms. The most important operations for producing covariants are the "transvection" processes, realized as certain bilinear differential operators acting on binary forms, or, more generally, analytic functions. According to the First Fundamental Theorem of classical invariant theory, all of the invariants and covariants for any system of polynomials or, more generally, functions, can be constructed through iterated transvectants and, in the inhomogeneous case, scaling processes. Thus, a proper grounding in these basic techniques is essential. Traditionally, such invariant processes are based on the symbolic method, which is the most powerful computational tool for computing and classifying invariants. However, no aspect of the classical theory has been as difficult to formalize or as contentious. The point of view taken here is nonstandard, relying on the construction of covariants and invariants as differential polynomials. Taking inspiration from work of Gel'fand and Dikii, [**77**], in solitons and the formal calculus of variations, I introduce a transform that mimics the Fourier transform of classical analysis and maps differential polynomials into algebraic polynomials. The transform is, in essence, the symbolic method realized in a completely natural manner, applicable equally well to polynomials and more general functions. The chapter concludes with proofs of the First Fundamental Theorem, which states that every covariant has symbolic form given by a polynomial in certain "bracket factors", and the Second Fundamental Theorem, which completely classifies the syzygies among the brackets. Although the determination of a complete Hilbert basis for the covariants of a general binary form turns out to be an extremely difficult problem, which has been solved only for forms of low degree, I shall prove a result due to Stroh and Hilbert that constructs an explicit rational basis for a form of arbitrary degree.

Chapter 7 introduces a graphical version of the symbolic method that can be used to simply and pictorially analyze complicated invariant-theoretic identities for binary forms. Each symbolic expression has an equivalent directed graph, or "digraph" counterpart, whereby algebraic identities among the symbolic forms translate into certain graphical operations that bear much similarity to basic operations in knot theory, [**124**], and thereby lead to a significant simplification with visual appeal. As an application, I show how to implement Gordan's method for constructing a complete system of fundamental invariants and covariants for binary forms, illustrated by the cubic and quartic examples.

At this point, we have covered the classical algebraic techniques underlying the theory of binary forms. Since the group of linear/projective transformations depends analytically on parameters, it is an example of a Lie transformation group. The theory of Lie groups includes a wide range of powerful calculus-based tools for the analysis of their invariants. Chapter 8 begins with a very brief introduction to Lie groups, including the general Frobenius Theorem that completely determines the local structure of the orbits and the fundamental invariants for regular actions. Here, invariants are classified up to functional dependence, rather than polynomial or rational dependence as was done in the more algebraic aspects of the theory; the number of fundamental invariants depends solely on the dimension of the group orbits. Even better, there is an explicit computation algorithm, which relies just on the Implicit Function Theorem, for constructing the invariants of regular Lie group actions. This method, known as "normalization", has its origins in Élie Cartan's theory of moving frames, [**33, 93**], which was developed for studying the geometry of curves and surfaces. Surprisingly, the normalization method has not been developed at all in the standard literature; the construction relies on a new theory of moving frames for general transformation group actions recently established by the author in collaboration with Mark Fels, [**69, 70**]. Applications to the classification of joint invariants and differential invariants for interesting transformation groups are provided.

In the theory of moving frames, the determination of symmetries, the complete solution to the equivalence problem, and the construction of canonical forms rely on the analysis of suitable differential invariants. In the case of planar curves, there is a single basic differential invariant — the group-theoretic *curvature* — along with a group-invariant *arc length* element. Higher order differential invariants are obtained by repeatedly differentiating curvature with respect to arc length. The first two fundamental differential invariants trace out the *signature set* which uniquely characterizes the curve up to group transformations. A direct application of the moving frame method leads to a remarkable theorem that the equivalence and symmetry of a binary form relies on merely *two* classical rational covariants! This result, first established in [**167**], reduces the entire complicated algebraic Hilbert basis to a simple pair of rational covariants whose functional dependencies completely encode the geometric properties of the binary form. I present a number of striking new consequences of this result, including a new bound on the number of discrete symmetries of polynomials. These innovative techniques are of much wider applicability and clearly deserve further development in the multivariate context.

While Chapter 8 develops "finite" Lie theory, the following chapter is concerned with Lie's powerful infinitesimal approach to invariance. Each Lie-theoretic object has an infinitesimal counterpart, and the replacement of complicated group-theoretic conditions by their infinitesimal analogs typically linearizes and significantly simplifies the analysis. The infinitesimal version of a Lie group is known as a Lie algebra, which contains the infinitesimal generators of the group action, realized as first order differential operators (or vector fields). Assuming connectivity, a function is invariant under the group if and only if it is annihilated by the infinitesimal generators, allowing methods from the theory of partial differential equations to be applied to the analysis of invariants. In the context of binary forms, the infinitesimal generators were, in fact, first recognized by Cayley, [**41**], to play an important role in the theory. I show how one can use these to build up general invariants from simpler "semi-" and "isobaric" invariants through an inductive procedure based on invariance under subgroups. The chapter culminates in a proof of the Hilbert Basis Theorem that relies on a particular differential operator that converts functions into invariants.

The final chapter is included to provide the reader with an orientation to pursue various generalizations of the basic methods and theories to multivariate polynomials and functions. Unfortunately, space has finally caught up with us at this point, and so the treatment is more superficial. Nevertheless, I hope that the reader will be sufficiently motivated to pursue the subject in more depth.

I have tried to keep the prerequisites to a minimum, so that the text can be profitably read by anyone trained in just the most standard undergraduate material. Certainly one should be familiar with basic linear algebra: vectors, matrices, linear transformations, Jordan canonical form, norms, and inner products — all of which can be found in any comprehensive undergraduate linear algebra textbook. Occasionally, I employ the tensor product construction. No knowledge of the general theory of polynomial equations is assumed. An introductory course in group theory could prove helpful to the novice but is by no means essential since I develop the theory of groups and their representations from scratch. All constructions take place over the real or complex numbers, and so no knowledge of more general field theory is ever required. One certainly does not need to take an abstract algebra course before starting; indeed, this text may serve as a good motivation or supplement for such a course!

In Chapters 8 and 9, I rely on multivariable differential calculus, at least as far as the Implicit Function Theorem, and the basic theory of first order systems of ordinary differential equations. In particular, the reader should be familiar with the solution to linear systems of differential equations, including matrix exponentials and their computation via Jordan canonical forms. I do not require any experience with Lie group theory or differential geometry, although the reader may wish to consult a basic text on manifolds, vector fields, and Lie groups to supplement the rather brief exposition here. (Chapter 1 of my own book [**168**] is particularly recommended!) Some of the more difficult results are stated without proof, although ample references are provided. I should remark that although the transform method adopted in Chapter 6 is inspired by the Fourier transform, no actual knowledge of the analytical Fourier theory is required.

Inevitably, the writing of an introductory text of moderate size requires making tough choices on what to include and what to leave out. Some of my choices are unorthodox. (Of course, if all choices were "orthodox", then there wouldn't be much point writing the book, as it would be a mere reworking of what has come before.) The most orthodox choice, followed in all the classical works as well as most modern introductions, is to concentrate almost entirely on the relatively modest realm of binary forms, relegating the vast hinterlands of multivariate polynomials and functions to an all too brief final chapter that cannot possibly do them justice. Of course, one motivation for this tactic is that most of the interesting explicit results and methods already make their appearance in the binary form case. Still, one tends to leave with the wish that such authors (including the present one) had more to say of substance in the multidimensional context.

Less orthodox choices include the reliance on calculus — differential operators, differential equations, differential invariants — as a framework for the general theory. Here we are in good company with the classics — Clebsch, [**49**], Gordan, [**89**], Grace and Young, [**92**], and even Hilbert, [**107**]. Post–Noetherian algebraists will no doubt become alarmed that I have regressed, in that the calculus-based tools are only valid in characteristic zero, or, more specifically, for the real and complex numbers, while "true" invariant theory requires that all fields be treated as equals, which means throwing out such "antiquated" analytical tools. My reply (and I speak here as the semi-applied mathematician I am) is that the primary physical and geometrical applications of invariant theory, which, after all, motivated its development, remain either real or complex, and it is here that much of the depth, beauty, and utility of the subject still resides. Another, more provocative, response is that the more interesting generalization of the classical techniques is not necessarily to fields of nonzero characteristic, but rather to more general associative and non-associative algebras, starting with the quaternions, octonions, Clifford algebras, quantum groups, and so on. One retains calculus (the quaternion calculus is a particularly pretty case) but gives up commutativity (and even possibly associativity). The development of a non-commutative classical invariant theory remains, as far as I know, completely unexplored.

The most original inclusion is the application of the Cartan theory of moving frames to the determination of symmetries and a solution to the equivalence problem for binary forms. Most of the constructions and results in this part of the text are new but can be readily comprehended by an advanced undergraduate student. This connection between geometry and algebra, I believe, opens up new and extremely promising vistas in both subjects — not to mention the connections with computer vision and image processing that served as one of my original motivations.

An unorthodox omission is the combinatorial and enumerative techniques that receive a large amount of attention in most standard texts. This was a difficult decision, and a topic I really did want to include. However, as the length of the manuscript crept up and up, it became clear that something had to go, and I decided this was it. The combinatorial formulae that count the number of invariants, particularly those based on Hilbert and Molien series and their generating functions, are very pretty and well worth knowing; see [**200**, **204**], for instance. However, as far as practical considerations go, they merely serve as indicators of what to expect and are of less help in the actual determination and classification of invariants. Indeed, in all the examples presented here, enumeration formulae are never used, and so their omission will not leave any gaps in the exposition. But the reader is well advised to consult other sources to rectify this omission.

The text is designed for the active reader. As always, one cannot learn mathematics by merely reading or attending lectures — one needs to *do* mathematics in order to absorb it. Thus, a large number and variety of exercises, of varying degrees of difficulty, are liberally interspersed throughout the text. They either illustrate the general theory with additional interesting examples or supply further theoretical results of importance that are left for the reader to verify. The student is strongly encouraged to attempt most exercises while studying the material.

I have also included many references and remarks of historical and cultural interest. I am convinced that one cannot learn a mathematical subject without being at least partially conversant with its roots and its original texts. Modern reformulations of classical mathematics, while sometimes (but not always) more digestible to the contemporary palate, often shortchange the contributions of the original masters. Worse yet, such rewritings can actually be harder for the novice to digest, since they tend to omit the underlying motivations or significance of the results and their interconnections with other parts of mathematics and applications. I am a firm believer in the need for a definite historical consciousness in mathematics. There is no better way of learning a theorem or construction than by going back to the original source, and a text (even at an introductory level) should make significant efforts to uncover and list where the significant ideas were conceived and brought to maturation. On the other hand, I do not pretend that my list of references is in any sense complete (indeed, the sheer volume of the nineteenth-century literature precludes almost any attempt at completeness); nevertheless, it includes many obscure but vital papers that clearly deserve a wider audience. I hope the reader is inspired to continue these historical and developmental studies in more depth.

The text has been typeset using the author's own OTEX system of macros. Details and software can be found at my web site:

`http://www.math.umn.edu/~olver` .

The figures were drawn with the aid of MATHEMATICA. Comments, corrections, and questions directed to the author are most welcome.

A Brief History

Classical invariant theory's origins are to be found in the early-nineteenth-century investigations by Boole, [**24**], into polynomial equations. The subject was nurtured by that indefatigable computer Cayley, to whom we owe many of the fundamental algorithms. Any reader of Cayley's collected works, [**36**], which include page after page of extensive explicit tables, cannot but be in awe of his computational stamina. (I often wonder what he might have accomplished with a functioning computer algebra system!) While the British school, led by Cayley and the flamboyant Sylvester, joined by Hermite in France, was the first to plow the virgin land, the actual flowering and maturation of the theory passed over to the Germans. The first wave of German experts includes Aronhold, the progenitor of the mystical "symbolic method", Clebsch, whose contributions metamorphosed into basic formulae in representation theory with profound consequences for quantum physics, and, most prominently, Gordan, the first among equals. Gordan's crowning achievement was his computational procedure and proof of the fundamental Basis Theorem that guarantees only a finite number of independent invariants for any univariate polynomial. The classical references by Clebsch, [**49**], Faà di Bruno, [**67**], and Gordan, [**89**], describe the resulting invariant theory of binary forms. A very extensive history of the nineteenth-century invariant theory, including copious references, was written by F. Meyer, [**151**]. Modern historical studies by Fisher, [**72**], and Crilly, [**55**], also document the underlying sociological and cultural implications of its remarkable history.

Despite much effort, extending Gordan's result to polynomials in two or more variables proved too difficult, until, in a profound stroke of genius, David Hilbert dramatically unveiled his general Basis Theorem in 1890. Hilbert's first, existential proof has, of course, had an incomparable impact, not just in classical invariant theory, but in all of mathematics, since it opened the door to the abstract algebraic approach that has characterized a large fraction of twentieth-century mathematics. Its immediate impact was the discreditation of the once dominant computational approach, which gradually fell into disrepute. Only in recent years, with the advent of powerful computer algebra systems and a host of new applications, has the computational approach to invariant theory witnessed a revival.

Nevertheless, the dawn of the twentieth century saw the subject in full florescence, as described in the marvelous (and recently translated) lectures of Hilbert, [**107**]. The texts by Grace and Young, [**92**], and Elliott, [**65**], present the state of the computational art, while Weitzenböck, [**229**], reformulates the subject under the guiding light of the new physics and tensor analysis. So the popular version of history, while appealing in its drama, is not entirely correct; Hilbert's paper did not immediately kill the subject, but rather acted as a progressive illness, beginning with an initial shock, and slowly consuming the computational body of the theory from within, so that by the early 1920's the subject was clearly moribund. Abstraction ruled: the disciples of Emmy Noether, a student of Gordan, led the fight against the discredited computational empire, perhaps as a reaction to Noether's original, onerous thesis topic that involved computing the invariants for a quartic form in three variables.

Although the classical heritage had vanished from the scene by mid-century, all was not quiet. The profoundly influential, yet often frustratingly difficult, book by Weyl, [**231**], places the classical theory within a much more general framework; polynomials now become particular types of tensorial objects, while, motivated by simultaneous developments in algebra and physics, the action of linear or linear fractional transformations is now extended to the vast realm of group representations. Attempts to reconcile both the classical heritage and Weyl's viewpoint with modern algebra and geometry have served to inspire a new generation of invariant theorists. Among the most influential has been Mumford's far-reaching development of the incisive methods of Hilbert, leading to the deep but abstract geometrical invariant theory, [**156**]. New directions, inspired by recent developments in representation theory and physics, appear in the recent work of Howe, [**112**]. Particular mention must be made of Rota and his disciples, [**94**, **135**], whose efforts to place the less than rigorous classical theory, particularly the symbolic method, on a firm theoretical foundation have had significant influence. The comprehensive text of Gurevich, [**97**], is a particularly useful source, which helped inspire a vigorous, new Russian school of invariant theorists, led by Popov, [**181**], and Vinberg, [**226**], who have pushed the theory into fertile new areas.

Of course, one cannot fail to mention the rise of modern computer algebra. Even the masters of the last century became stymied by the sheer complexity of the algebraic formulas and manipulations that the subject breeds. The theory of Gröbner bases, cf. [**54**], has breathed new life into the computational aspect of the subject. Sturmfels' elegant book, [**204**], gives an excellent survey of current work in this direction and is particularly recommended to the student wishing to continue beyond the material covered here. The influence of classical invariant theory can be felt throughout mathematics and extends to significant physical applications, ranging from algebra and number theory, [**79**], through combinatorics, [**201**], Riemannian geometry, [**149**, **150**, **229**], algebraic topology, [**196**], and ordinary differential equations, [**235**, **195**]. Applications include continuum mechanics, [**197**], dynamical systems, [**195**], engineering systems and control theory, [**213**], atomic physics, [**189**], and even computer vision and image processing, [**157**]. This text should prove to be useful to students in all of these areas and many more.

Acknowledgments

Many people deserve thanks for helping inspire my interest in this subject and desire to put pen to paper (or, more accurately, finger to keyboard). They include John Ball, whose questions in nonlinear elasticity directly motivated my initial forays; Gian–Carlo Rota, whose wonderful lectures opened my vistas; and Bernd Sturmfels, who patiently introduced me to modern computational tools. I would particularly like to thank my student Irina Berchenko for her careful proofreading of the manuscript. Most of all, I must express my heartfelt gratitude to my wonderful wife, Cheri Shakiban, who directly collaborated with me on several invariant theoretic papers, [**171**, **172**], which form the basis of significant parts of this text. I am incredibly fortunate that she has been such a major part of my life.

I have dedicated this book to my parents. My father, Frank W.J. Olver, is a great applied analyst and certainly played a direct, inspirational role in my choice of career. I wish my mother, Grace E. Olver, were still alive to see the further fruits of her love and care. They both set me on those important first steps in mathematics, and for this I am eternally grateful.

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