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Hugh Neill and Douglas Quadling
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Module FP2

Further Pure 2

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1 Differentiating inverse trigonometric functions

Throughout the course you have gradually been increasing the number of functions that you can differentiate and integrate. This chapter extends this development to inverse trigonometric functions. When you have completed it, you should

- know the derivatives of $\tan^{-1} x$, $\sin^{-1} x$ and $\cos^{-1} x$
- know the integrals corresponding to these derivatives
- be familiar with other inverse trigonometric functions and relations between them
- use these relations to differentiate other inverse trigonometric functions.

1.1 The inverse tangent

The simplest of the inverse trigonometric functions to differentiate is $\tan^{-1} x$. You can do this directly from the definition, that $y = \tan^{-1} x$ is the number such that

$$\tan y = x \quad \text{and} \quad -\tfrac{1}{2}\pi < y < \tfrac{1}{2}\pi.$$

You know, from a general result about inverse functions (see C3 Section 2.9) that its graph is the reflection in the line $y = x$ of the part of the graph of $y = \tan x$ for which $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$.

This is shown in Fig. 1.1.

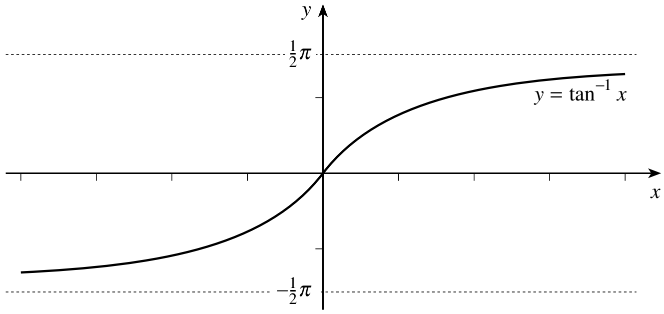

Fig. 1.1

The equation $\tan y = x$ is not in the form $y = \ldots$, but it can be differentiated using the method for curves defined implicitly described in C4 Chapter 8. The derivative with respect to x of $\tan y$ is $\sec^2 y \frac{dy}{dx}$, and the derivative of x is 1, so

$$\tan y = x \quad \Rightarrow \quad \sec^2 y \frac{dy}{dx} = 1$$
$$\Leftrightarrow \quad \frac{dy}{dx} = \frac{1}{\sec^2 y}.$$

But this isn't really satisfactory. When you differentiate $y = \tan^{-1} x$, you expect an answer in terms of x, not y. However, this is easily dealt with. Since $\sec^2 y = 1 + \tan^2 y$, and $\tan y = x$,

$$\sec^2 y = 1 + x^2,$$

so $$\frac{dy}{dx} = \frac{1}{1+x^2}.$$

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}.$$

It is interesting that the derivative of $\tan^{-1} x$ is not any sort of trigonometric function, but a rational function. This may remind you of what happens with $\ln x$, whose derivative $\frac{1}{x}$ doesn't involve logarithms or exponentials.

Notice that, since $1 + x^2 \geqslant 1$, the gradient of the graph in Fig. 1.1 is less than or equal to 1 throughout its length.

Example 1.1.1
Differentiate with respect to x (a) $\tan^{-1} \frac{1}{3}x$, (b) $\tan^{-1} x^2$.

Both derivatives can be found by using the chain rule.

(a) $$\frac{d}{dx}\tan^{-1}\tfrac{1}{3}x = \frac{1}{1+\left(\frac{1}{3}x\right)^2} \times \tfrac{1}{3} = \frac{\frac{1}{3}}{1+\frac{1}{9}x^2}.$$

To write this more simply, multiply top and bottom of the fraction by 9, to get

$$\frac{d}{dx}\tan^{-1}\tfrac{1}{3}x = \frac{9 \times \frac{1}{3}}{9\left(1+\frac{1}{9}x^2\right)} = \frac{3}{9+x^2}.$$

(b) $$\frac{d}{dx}\tan^{-1} x^2 = \frac{1}{1+(x^2)^2} \times 2x = \frac{2x}{1+x^4}.$$

Example 1.1.2
Find $\int \tan^{-1} x \, dx$

In C4 Example 2.1.3, $\int \ln x \, dx$ was found by writing the integrand as $\ln x \times 1$ and using integration by parts. This works because the derivative of $\ln x$ is a rational function of x, and doesn't involve a logarithm.

You can find $\int \tan^{-1} x \, dx$ in a similar way, and for the same reason.

Writing $u = \tan^{-1} x$ and $\frac{dv}{dx} = 1$, so that $v = x$,

$$\int \tan^{-1} x \, dx = \tan^{-1} x \times x - \int \frac{1}{1+x^2} \times x \, dx.$$

You should recognise this last integral, $\int \frac{x}{1+x^2}\,dx$, as a constant multiple of the form $\int \frac{f'(x)}{f(x)}\,dx$, which is $\ln|f(x)| + k$ (see C4 Section 2.4). So

$$\int \frac{x}{1+x^2}\,dx = \tfrac{1}{2}\int \frac{2x}{1+x^2}\,dx = \tfrac{1}{2}\ln(1+x^2) + k.$$

So $$\int \tan^{-1} x\,dx = x\tan^{-1} x - \tfrac{1}{2}\ln(1+x^2) - k.$$

1.2 Inverse sine and cosine

The method of differentiating $\sin^{-1} x$ and $\cos^{-1} x$ is similar to that for $\tan^{-1} x$, but there are some small complications. The easier and more important is $\sin^{-1} x$, so begin with this.

The definition is that $y = \sin^{-1} x$ is the number such that

$$\sin y = x \quad \text{and} \quad -\tfrac{1}{2}\pi \leqslant y \leqslant \tfrac{1}{2}\pi.$$

Its graph is shown in Fig. 1.2. The domain of the function is $-1 \leqslant x \leqslant 1$.

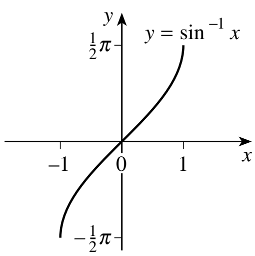

Fig. 1.2

Differentiating this equation by the implicit method gives

$$\sin y = x \quad \Rightarrow \quad \cos y\frac{dy}{dx} = 1$$
$$\Leftrightarrow \quad \frac{dy}{dx} = \frac{1}{\cos y}.$$

Again this derivative has to be expressed in terms of x, and this time the relation you want is $\cos^2 y + \sin^2 y = 1$, so that $\cos y = \pm\sqrt{1-\sin^2 y}$, which is $\pm\sqrt{1-x^2}$. But should the sign be + or −?

To answer this, look at the graph in Fig. 1.2. You can see that the gradient of the graph is always positive, so the + sign must be chosen. So replacing $\cos y$ by $\sqrt{1-x^2}$,

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

Example 1.2.1
Find the domains, and the derivatives with respect to x, of

(a) $\sin^{-1}\frac{1}{5}x$, (b) $\sin^{-1}(1-x)$.

(a) The domain of $\sin^{-1}x$ is $-1 \leqslant x \leqslant 1$, so the numbers in the domain of $\sin^{-1}\frac{1}{5}x$ must satisfy the inequalities $-1 \leqslant \frac{1}{5}x \leqslant 1$. The domain is therefore $-5 \leqslant x \leqslant 5$.

Using the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}\tfrac{1}{5}x = \frac{1}{\sqrt{1-\left(\frac{1}{5}x\right)^2}} \times \tfrac{1}{5}$$
$$= \frac{1}{5\sqrt{1-\frac{1}{25}x^2}}$$
$$= \frac{1}{\sqrt{25\left(1-\frac{1}{25}x^2\right)}}$$
$$= \frac{1}{\sqrt{25-x^2}}.$$

(b) The numbers in the domain of $\sin^{-1}(1-x)$ must satisfy the inequalities $-1 \leqslant 1-x$ and $1-x \leqslant 1$, that is $x \leqslant 2$ and $x \geqslant 0$. The domain is therefore $0 \leqslant x \leqslant 2$.

Using the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}(1-x) = \frac{1}{\sqrt{1-(1-x)^2}} \times (-1)$$
$$= \frac{-1}{\sqrt{1-(1-2x+x^2)}}$$
$$= \frac{-1}{\sqrt{2x-x^2}}.$$

You can find the derivative of $\cos^{-1}x$ using the same method as for $\sin^{-1}x$. This is left for you to do for yourself in Exercise 1A Question 7. But there is an even easier way.

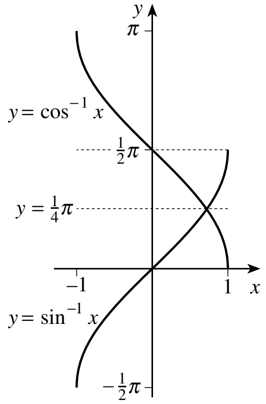

Fig. 1.3

Figure 1.3 shows the graphs of $y = \sin^{-1} x$ and $y = \cos^{-1} x$ drawn using the same axes. You can see that the graphs are reflections of each other in the line $y = \frac{1}{4}\pi$. So, for each value of x, the gradient of $y = \cos^{-1} x$ is minus the gradient of $y = \sin^{-1} x$. That is,

$$\frac{d}{dx}\cos^{-1} x = -\frac{d}{dx}\sin^{-1} x = -\frac{1}{\sqrt{1-x^2}}.$$

If $-1 < x < 1$,

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \quad \text{and} \quad \frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}.$$

The derivatives of $\tan^{-1} x$, $\sin^{-1} x$ and $\cos^{-1} x$ are important and you should remember them.

You may be surprised that the derivatives of $\sin^{-1} x$ and $\cos^{-1} x$ are stated for $-1 < x < 1$, and not for the whole domain $-1 \leqslant x \leqslant 1$. It is easy to see why from Fig. 1.3. When $x = -1$ and $x = 1$ the tangents to the graphs are parallel to the y-axis, so that the gradient is undefined. Also of course $\frac{1}{\sqrt{1-x^2}}$ has no meaning for these values of x.

Exercise 1A

1 Find the derivatives of the following with respect to x.

(a) $\tan^{-1} 2x$ (b) $\sin^{-1} \frac{1}{3}x$ (c) $x \tan^{-1} x$ (d) $(\sin^{-1} x)^2$

(e) $\sin^{-1}\sqrt{x}$ (f) $\tan^{-1}(x\sqrt{x})$ (g) $\sin^{-1}\sqrt{1-x^2}$

2 Find the minimum point of the graph of $y = x^2 - 4\tan^{-1} x$.

3 State the natural domain (that is, the largest possible domain) of the function $f(x) = 5x - 3\sin^{-1} x$. Find the turning points on the graph of $y = f(x)$, and sketch the graph. Hence find the range of the function.

4 Repeat Question 3 for the function $f(x) = 5x - 4\sin^{-1} x$.

5 The tangent to $y = (\tan^{-1} x)^2$ at the point where $x = 1$ cuts the y-axis at the point P. Find the y-coordinate of P.

6 Find the maximum value of $f(x) = (\sin^{-1} x)^2 \cos^{-1} x$ in the interval $0 < x < 1$.

7 Find $\frac{d}{dx}\cos^{-1} x$ by the method used in Section 1.2 to find $\frac{d}{dx}\sin^{-1} x$.

1.3 The corresponding integrals

The derivatives in Sections 1.1 and 1.2 can also be interpreted as integrals. From $\dfrac{d}{dx}\tan^{-1}x = \dfrac{1}{1+x^2}$ it follows that

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + k.$$

And from $\dfrac{d}{dx}\sin^{-1}x = \dfrac{1}{\sqrt{1-x^2}}$ you can deduce

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + k.$$

You could also use $\dfrac{d}{dx}\cos^{-1}x = -\dfrac{1}{\sqrt{1-x^2}}$ to obtain $\displaystyle\int \frac{1}{\sqrt{1-x^2}}dx = -\cos^{-1}x + k'$, but there is no point in using two different forms for the same integral, and $\sin^{-1}x + k$ is simpler. What is the connection between k and k'?

Example 1.3.1

Find $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx$.

$$\begin{aligned}\int_0^1 \frac{1}{1+x^2}\,dx &= \left[\tan^{-1}x\right]_0^1 \\ &= \tan^{-1}1 - \tan^{-1}0 \\ &= \tfrac{1}{4}\pi - 0 \\ &= \tfrac{1}{4}\pi.\end{aligned}$$

Use a graphic calculator to display the graph of $y = \dfrac{1}{1+x^2}$ for $0 \leqslant x \leqslant 1$, and identify the area represented by the integral in Example 1.3.1. It is interesting that the number π appears in calculating an area which has nothing to do with a circle.

Example 1.3.2

Find $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$.

This integral needs a little more care. Notice that the integrand $\dfrac{1}{\sqrt{1-x^2}}$ is not defined when $x = 1$, because $\sqrt{1-1^2} = 0$. So this is an improper integral, and it must be calculated as a limit.

Use a graphic calculator to display the graph of $y = \dfrac{1}{\sqrt{1-x^2}}$.

Begin by finding, for a number s such that $0 < s < 1$,

$$\begin{aligned}\int_0^s \frac{1}{\sqrt{1-x^2}}\,dx &= \left[\sin^{-1} x\right]_0^s \\ &= \sin^{-1} s - \sin^{-1} 0 \\ &= \sin^{-1} s.\end{aligned}$$

The graph of $y = \sin^{-1} x$ in Fig. 1.2 shows that, as $x \to 1$, $\sin^{-1} x \to \frac{1}{2}\pi$. So

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx &= \lim_{s\to 1} \int_0^s \frac{1}{\sqrt{1-x^2}}\,dx \\ &= \lim_{s\to 1}(\sin^{-1} s) \\ &= \tfrac{1}{2}\pi.\end{aligned}$$

You will find that you often want to find integrals like those at the beginning of this section in a slightly more general form, as $\int \frac{1}{a^2+x^2}\,dx$ or $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$, where a is a positive number. It is easy to do this by using the substitution $x = au$. Then $\frac{dx}{du} = a$, so

$$\begin{aligned}\int \frac{1}{a^2+x^2}\,dx &= \int \frac{1}{a^2+a^2u^2} \times a\,du \\ &= \int \frac{a}{a^2(1+u^2)}\,du \\ &= \frac{1}{a}\int \frac{1}{1+u^2}\,du \\ &= \frac{1}{a}\tan^{-1} u + k \\ &= \frac{1}{a}\tan^{-1}\frac{x}{a} + k,\end{aligned}$$

and

$$\begin{aligned}\int \frac{1}{\sqrt{a^2-x^2}}\,dx &= \int \frac{1}{\sqrt{a^2-a^2u^2}} \times a\,du \\ &= \int \frac{a}{\sqrt{a^2(1-u^2)}}\,du \\ &= \frac{a}{a}\int \frac{1}{\sqrt{1-u^2}}\,du \\ &= \sin^{-1} u + k \\ &= \sin^{-1}\frac{x}{a} + k.\end{aligned}$$

You will need to remember these results, either in the forms given at the beginning of the section or in these more general forms.

If $a > 0$,

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + k, \qquad \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + k.$$

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Example 1.3.3

Figure 1.4 shows the graph of $y = \frac{1}{\sqrt{4+x^2}}$ for $-2 \leqslant x \leqslant 2$. Find the volume of the solid formed when the region bounded by this curve and parts of the lines $x = -2$, $x = 2$ and the x-axis is rotated though a complete revolution about the x-axis.

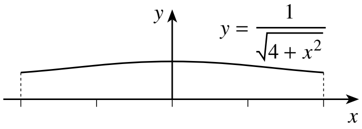

Fig. 1.4

The volume is given by the integral

$$\begin{aligned}\int_{-2}^{2} \pi y^2 \, dx &= \pi \int_{-2}^{2} \frac{1}{4+x^2} \, dx \\ &= \pi \left[\tfrac{1}{2} \tan^{-1} \frac{x}{2}\right]_{-2}^{2} \\ &= \tfrac{1}{2}\pi (\tan^{-1} 1 - \tan^{-1}(-1)) \\ &= \tfrac{1}{2}\pi (\tfrac{1}{4}\pi - (-\tfrac{1}{4}\pi)) \\ &= \tfrac{1}{2}\pi \times \tfrac{1}{2}\pi = \tfrac{1}{4}\pi^2.\end{aligned}$$

The volume of the solid is $\frac{1}{4}\pi^2$.

Example 1.3.4

Find $\int \frac{1}{\sqrt{a^2 - b^2x^2}} \, dx$, where a and b are positive constants.

If bx is written as au, then $\sqrt{a^2 - b^2x^2}$ becomes $\sqrt{a^2 - a^2u^2}$, which simplifies to $a\sqrt{1-u^2}$. So, substituting $x = \frac{au}{b}$,

$$\begin{aligned}\int \frac{1}{\sqrt{a^2 - b^2x^2}} \, dx &= \int \frac{1}{a\sqrt{1-u^2}} \times \frac{a}{b} du = \frac{1}{b} \int \frac{1}{\sqrt{1-u^2}} \, du \\ &= \frac{1}{b} \sin^{-1} u + k = \frac{1}{b} \sin^{-1} \frac{bx}{a} + k.\end{aligned}$$

Example 1.3.5

Find $\int_{-1}^{1} \frac{1}{9x^2 + 6x + 5} \, dx$.

Since $9x^2 + 6x + 5 = (3x+1)^2 + 4$, substitute $3x + 1 = 2u$. Then $\frac{dx}{du} = \frac{2}{3}$; also when $x = -1$ and 1, $u = -1$ and 2 respectively. So the integral becomes

$$\begin{aligned}\int_{-1}^{2} \frac{1}{4u^2 + 4} \times \tfrac{2}{3} du &= \tfrac{1}{6} \int_{-1}^{2} \frac{1}{u^2 + 1} \, du = \tfrac{1}{6} \left[\tan^{-1} u\right]_{-1}^{2} \\ &= \tfrac{1}{6}(\tan^{-1} 2 - \tan^{-1}(-1)) = \tfrac{1}{6}(\tan^{-1} 2 + \tan^{-1} 1).\end{aligned}$$

If you want a numerical answer, don't forget to put your calculator into radian mode. The value is 0.315, correct to 3 decimal places.

Exercise 1B

1 Evaluate the following definite integrals. Give each answer as an exact multiple of π if possible; otherwise give the answer correct to 3 significant figures.

(a) $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{4-x^2}}\,dx$ (b) $\displaystyle\int_{0}^{5} \frac{1}{25+x^2}\,dx$ (c) $\displaystyle\int_{1}^{3} \frac{1}{4+x^2}\,dx$

(d) $\displaystyle\int_{1}^{3} \frac{1}{3+x^2}\,dx$ (e) $\displaystyle\int_{0}^{1} \frac{1}{\sqrt{2-x^2}}\,dx$ (f) $\displaystyle\int_{-4}^{-3} \frac{1}{\sqrt{25-x^2}}\,dx$

2 Find the following infinite integrals.

(a) $\displaystyle\int_{1}^{\infty} \frac{1}{1+x^2}\,dx$ (b) $\displaystyle\int_{0}^{\infty} \frac{1}{9+x^2}\,dx$ (c) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{100+x^2}\,dx$

3 Find the following improper integrals.

(a) $\displaystyle\int_{0}^{5} \frac{1}{\sqrt{25-x^2}}\,dx$ (b) $\displaystyle\int_{-3}^{3} \frac{1}{\sqrt{9-x^2}}\,dx$ (c) $\displaystyle\int_{1}^{2} \frac{1}{\sqrt{4-x^2}}\,dx$

4 Use a substitution of the form $x = cu$ for a suitable value of c to find the following indefinite integrals.

(a) $\displaystyle\int \frac{1}{9+4x^2}\,dx$ (b) $\displaystyle\int \frac{1}{\sqrt{4-9x^2}}\,dx$ (c) $\displaystyle\int \frac{1}{\sqrt{1-4x^2}}\,dx$

(d) $\displaystyle\int \frac{1}{1+9x^2}\,dx$ (e) $\displaystyle\int \frac{1}{2+3x^2}\,dx$ (f) $\displaystyle\int \frac{1}{\sqrt{4-5x^2}}\,dx$

5 By completing the square and then using a substitution of the form $x = a + bu$, find the following indefinite integrals.

(a) $\displaystyle\int \frac{1}{x^2+2x+2}\,dx$ (b) $\displaystyle\int \frac{1}{x^2+6x+13}\,dx$ (c) $\displaystyle\int \frac{1}{4x^2-12x+25}\,dx$

(d) $\displaystyle\int \frac{1}{\sqrt{5-4x-x^2}}\,dx$ (e) $\displaystyle\int \frac{1}{\sqrt{8+6x-9x^2}}\,dx$ (f) $\displaystyle\int \frac{1}{\sqrt{10x-x^2}}\,dx$

6 Evaluate the following definite integrals. Give your answers to 3 significant figures. In some parts you may need to use one of the methods described in Question 4 and Question 5.

(a) $\displaystyle\int_{0}^{2} \frac{1}{x^2+25}\,dx$ (b) $\displaystyle\int_{1}^{3} \frac{1}{1+16x^2}\,dx$ (c) $\displaystyle\int_{-1}^{1} \frac{1}{x^2-6x+25}\,dx$

(d) $\displaystyle\int_{0}^{\infty} \frac{1}{9+25x^2}\,dx$ (e) $\displaystyle\int_{1}^{2} \frac{1}{\sqrt{9-x^2}}\,dx$ (f) $\displaystyle\int_{0}^{1} \frac{1}{\sqrt{16-9x^2}}\,dx$

(g) $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{3+2x-x^2}}\,dx$ (h) $\displaystyle\int_{-0.5}^{0.5} \frac{1}{\sqrt{1-4x^2}}\,dx$ (i) $\displaystyle\int_{1}^{2} \frac{1}{\sqrt{4x-x^2}}\,dx$

7 Find $\displaystyle\int \frac{1}{a^2+b^2x^2}\,dx$, where a and b are positive constants.

8 Show that the rule $\displaystyle\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + k$ remains true if $a < 0$, but that $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + k$ doesn't.