The Independent University of Moscow and Student Sessions at the IUM

Student Sessions at the Independent University of Moscow is a new tradition in the mathematical life of Moscow. The Independent University (briefly the IUM) itself is a young child of the new times in Russia. In this introduction, a brief description of the IUM is presented.

The history of the IUM begins with a meeting held in the summer of 1991 at Moscow High School No. 57. The meeting was initiated by N. Konstantinov. According to his suggestion, the future team of the Independent University simply decided to start teaching university courses in mathematics, beginning in September 1991. The subsequent history of this meeting characterizes the historical period when it occurred. Had it taken place in Stalin's time, all the participants of the meeting would have been immediately arrested. Had it opened in Brezhnev's time, nothing would have resulted from the meeting. Since it happened in Gorbachev's time, it turned out to be the beginning of the history of the Independent University of Moscow. The founders of the Independent University instituted a small fund from which the IUM was supported during the first period of its work.

The founders were organized into the Scientific Council of the IUM, presided over by V. I. Arnold and consisting of A. A. Beilinson, the late R. L. Dobrushin, L. D. Faddeev, B. M. Feigin, Yu. S. Ilyashenko, A. G. Khovanskii, A. A. Kirillov, S. P. Novikov, A. N. Rudakov, M. A. Shubin, Ya. G. Sinai, and V. M. Tikhomirov. Professors P. Deligne and R. MacPherson, both of whom have actively supported the IUM since its foundation, are Honorary Members of the Scientific Council.

In the first years, the administration was carried out by N. Konstantinov with his students and friends working as assistants: S. Komarov was responsible for economic and financial matters, V. Imaikin prepared the lecture notes, M. Vyalyi organized the teaching process. During the first year, the IUM worked in the School of Informational Technologies near Moscow State University. During the next four academic years, Moscow High School No. 2 kindly invited the IUM to have classes in its building in the evenings. We are especially grateful to the director of the school, P. V. Khmilinskii, for his hospitality.

In 1994, the Prefect of the Central District of Moscow, A. Muzykantskii, proposed that we organize a new institution, related both to high school and university mathematics, to which a building might be officially presented by

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the authorities. The bureaucratic work needed for the functioning of this new institution and for solving numerous administrative problems related to getting the new building was enormous. We began to look for an executive director for this new institution who would be able to carry out this work. As I said to one of my older friends and colleagues, we needed a person who would be a professional in the administrative world and would understand our university ideals. “Don’t bother,” my friend answered, “such a person simply does not exist.” But we were lucky to find two people of the kind we were dreaming about: I. Yashchenko and V. Furin, both alumni of the Moscow State University. At that time, both had successful enterprises; in parallel, I. Yashchenko continued his mathematical research work.

Producing all the necessary documentation was a full time job, and in half a year it resulted in a gift from the Moscow government: in June 1995, the Major of Moscow, Yu. Luzhkov, signed the ordinance giving a new institution, the Moscow Center of Continuous Mathematical Education, an unfinished building in the historical center of Moscow. The IUM was required to find, by its own efforts, $1 000 000 needed to finish the construction of the building.

At that time, it was a brick four-story house without a roof, with unfinished staircases and floors covered by crushed bricks, like after a bombing. We then declared that we would find the necessary sum, having no concrete sources whatever in mind, only hoping that, for such a good enterprise, the money would eventually be found. Indeed, in August 1995, the Moscow government granted $1 500 000 for finishing the construction of the building and furnishing it, and in a year it was concluded, according to a project presented by the IUM team. On September 26, 1996, the inauguration ceremony of the new building took place, and two closely related institutions, the IUM and the MCCME, began to work in it.

Besides the support of the IUM, the MCCME carries on a lot of activities related to high school education: various mathematical olympiads, lectures for high school teachers, conferences dedicated to educational problems, and so on.

During the last twelve years, when the Moscow Mathematical Society, and later the MCCME, became directly involved in the organization of the famous Moscow Mathematical Olympiads, it regained and exceeded its former popularity. Last year, three thousand high school students participated in the Olympiad, and the number of awards equalled the total number of participants of the Moscow Olympiad of 1992.

Other activities of the MCCME include a conference on educational problems, organized in 2000. Beginning in 2001, the MCCME organizes an annual Summer School, which brings together high school and university students with lecturers of the highest level, academicians Anosov, Arnold, and the late Bolibrukh included.

The present status of the Independent University is as follows. The first President of the Independent University was M. Polivanov, a mathematical physicist and philosopher, who passed away a year after the beginning of his Presidency. The IUM has two colleges, the Higher College of Mathematics and the Higher College of Mathematical Physics. The former was first headed by A. Rudakov, and now it is headed by Yu. Ilyashenko; the latter was headed by O. Zavialov, now by A. I. Kirillov. We have about 100 students in both colleges and about 40 freshmen each year. The graduate school of the IUM was founded in 1993 as a result of the initiative of A. Beilinson, B. Feigin, and V. Ginzburg. Twenty seven people have graduated from this school and passed their Ph.D. theses as of now.

At present, most of our male students study in parallel at two universities, say Moscow State and the IUM, in order to have military draft exemption. Therefore, our classes take place in the evenings.

The IUM gives a chance to create their own mathematical schools to mathematicians not involved in the teaching process at Moscow State University. The seminars of B. Feigin, S. Natanzon, O. Sheinman, O. Shvartsman, M. Tsfasman, and V. Vassiliev have been continuing for several years at the IUM.

Lecture courses at the IUM were given by D. V. Anosov, V. I. Arnold, A. A. Kirillov, S. P. Novikov, Ya. G. Sinai, V. A. Vassiliev, A. A. Belavin, V. K. Beloshapka, B. M. Feigin, S. M. Gusein-Zade, Yu. S. Ilyashenko, A. G. Khovanskii, I. M. Krichever, A. N. Rudakov, A. G. Sergeev, V. M. Tikhomirov, M. A. Tsfasman, and many others. The courses of Arnold (PDE), Vassiliev (Topology), and Anosov (Dynamical Systems) were published as books later.

The IUM provides teaching possibilities to professors who have full time positions in the West now. They are realized in the form of crash courses, usually one month long but so intensive that they are equivalent to semester courses. Such courses were given by A. A. Kirillov, A. Khovanski, I. Krichever, A. Katok (who is a Foreign Member of the IUM faculty), P. Cartier, and D. Anosov. In 1995–96, A. Khovanski gave a regular course in honors calculus; he got permission to be on leave from Toronto University, where he had a full position at the time.

The IUM tries to be a place to which Russian mathematicians can return after their work abroad, if they will. At present, we have seven young faculty members who obtained their Ph.D. abroad but are now teaching at the IUM.

Beginning in 2001, the IUM launched a new periodical, the *Moscow Mathematical Journal*. Among the authors of the papers already published and presented are V. I. Arnold, P. Deligne, G. Faltings, V. Ginzburg, A. Givental, A. J. de Jong, A. and S. Katok, C. Kenig, A. Khovanski, A. A. Kirillov, Ya. Sinai, M. Tsfasman, A. Varchenko, D. Zagier, and many others.

In the spring of 2001, the IUM organized a Study Abroad Program, called Math in Moscow (MIM), for foreign students. They are invited to the IUM for one semester to take mathematical and nonmathematical courses and to plunge into Russian cultural life. The credits for these courses are transferable to North American and Canadian universities. Up to now, the MIM program was attended by students from Berkeley, Cornell, Harvard, MIT, McHill, universities of Montreal and Toronto, Penn State, and many others.

In order to support young researchers, the Möbius Competition for the best research work of undergraduate or graduate students was organized in 1997 and sponsored by V. Balikoev and A. Kokin, both alumni of the Moscow Institute of Mathematics and Electronics. The winners were A. Kuznetsov (1997), V. Timorin (1998), A. Bufetov (1999) (all from the IUM), S. Shadrin and A. Melnikhov (2000) (MSU), A. Ershler (2001) (St. Petersburg University), V. Kleptsyn and L. Rybnikov (2002), S. Chulkov (2003, first place), and S. Oblezin and S. Shadrin (2004, second place). Recently, thanks to the initiative of V. Kaloshin (Caltech) who raised extra funds, the number of stipends was increased from one to three, and the duration was extended from one to two years.

Last but not least, the IUM has organized Student Sessions, which were held beginning in 1997. The first lecture was delivered by Arnold, one of our Founding Fathers, President of the Scientific Council of the IUM. The lectures given in 1998–2000 are presented to the reader. The lectures were intended for a large audience, from students to professional researchers. They contained no proofs or technical details. The objective was to give panoramas of whole research areas and describe new ideas.

Beginning in 2001, the Sessions were transformed into a regular mathematics research seminar, called Globus. This seminar brings together mathematicians from all sides of Moscow. It is in a sense parallel to the sessions of the Moscow Mathematical Society and intended for a similar audience. The lectures are taped and collected into volumes. Two volumes of these lectures will appear in Russian soon.

Of course, numerically the IUM plays a negligible role in Russian cultural life, but its influence, in my opinion, is far from negligible. It may be characterized by a quotation from the Gospel:

> *The Kingdom of Heaven is like unto leaven, that a women took and hit into three measures of meal till the whole was leavened.* (Mt, 13:33)

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The lectures at the Student Sessions and later at the Globus seminars were tape recorded. Then these records were decoded and edited by Professor V. Prasolov, translated into English, and sent to the authors to make final corrections. It is a hard job to transform speech into written text. This volume, as well as the subsequent ones prepared for publication in Russian, would never have appeared without the energy and devotion of V. Prasolov. The organizers of the Student Sessions, as well as of the Globus seminars, are cordially grateful to him.

V. I. Arnold

Mysterious mathematical trinities

Lecture on May 21, 1997

I shall try to tell about some phenomena in mathematics that make me surprised. In most cases, they are not formalized. They cannot even be formulated as conjectures. A conjecture differs in that it can be disproved; it is either true or false.

We shall consider certain observations that lead to numerous theorems and conjectures, which can be proved or disproved. But these observations are most interesting when considered from a general point of view.

I shall explain this general point of view for a simple example from linear algebra.

The theory of linear operators is described in modern mathematics as the theory of Lie algebras of series A_n, i.e., $\mathfrak{sl}(n+1)$, and formulated in terms of root systems. A root system can be assigned to any Coxeter group, that is, a finite group generated by reflections (at least, to any crystallographic group). If we take a statement of linear algebra which refers to this special case of the group A_n and remove all the content from its formulation, so as to banish all mentions of eigenvalues and eigenvectors and retain only roots, we will obtain something that can be applied to the other series, B_n, C_n, and D_n, including the exceptional ones E_6, E_7, E_8, F_4, and G_2 (and, sometimes, even to all the Coxeter systems, including the noncrystallographic symmetry groups of polygons, of the icosahedron, and of the hypericosahedron, which lives is four-dimensional space).

From this point of view, the geometries of other series (B, C, ...) are not geometries of vector spaces with additional structures, such as Euclidean, symplectic, etc. (although formally, they, of course, are); they are not daughters of A-geometry but its sisters enjoying equal rights.

The above classification of simple Lie algebras, which is due to Killing (and, hence, attributed to Cartan), has an infinite-dimensional analogue – in analysis. The algebraic problem solved by Killing, Cartan, and Coxeter has an infinite-dimensional analogue in the theory of Lie algebras of diffeomorphism groups. Given a manifold M, the group $\mathrm{Diff}(M)$ of all diffeomorphisms of M naturally arises. This group (more precisely, the connected component of the identity element in this group) is algebraically simple, i.e., it has no normal divisors. There exist other similar "simple" theories, which resemble the geometry of manifolds but differ from it. They were also classified by Cartan at one time.[1] Having imposed a few fairly natural constraints, he discovered that there exist six series of such groups:

Diff(M);

SDiff(M), the group of diffeomorphisms preserving a given form of volume;

SpDiff(M, ω^2), the group of symplectomorphisms.

Next, there are complex manifolds and groups of holomorphic diffeomorphisms.

There is also the very important contact group, the group of contactomorphisms.

Finally, there are conformal versions of some of these theories. I shall not describe them in detail.

The idea which I mentioned is that, in these theories, there is something similar to the passage from theorems of linear algebra, i.e., from the A_n root system, to other root systems. In other words, in the whole of mathematics (at least, of the geometry of manifolds), there are higher-level operations (e.g., symplectization) that assign analogues from the theory of manifolds with volume elements or of symplectic manifolds to each definition and each theorem of manifold theory. This is by no means a rigorous statement; such an operation is not a true functor.

For example, an element of the Lie algebra of the diffeomorphism group is a vector field. The symplectization of a vector field is a Hamiltonian field determining the Hamilton equation

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}.$$

Other situations are more involved. It is difficult to understand what becomes of notions of linear algebra under the passage to other geometries. But even dealing with the symplectomorphism group, we take only the vector fields that are determined by a single-valued Hamiltonian function rather than the entire Lie algebra of this group. These vector fields form the commutator of the Lie algebra, which does not coincide with the entire Lie algebra of the symplectomorphism group. Still, when we are able to find regular analogues for some notions of some geometry in another geometry, the reward is very significant.

Consider two examples.

1. Symplectization. So-called Arnold's conjectures (1965) about fixed points of symplectomorphisms were stated in an attempt to symplectize the Poincaré–Euler theorem that the sum of indices of the singular points of a vector field

[1] See, in particular, E. Cartan. *Selected Works* (Moscow: MTsMNO, 1998) [in Russian]. (*Editor's note*)

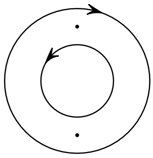

Figure 1

on a manifold is equal to the Euler characteristic. They estimate the number of closed trajectories for Hamiltonian vector fields by means of the Morse inequalities (i.e., in terms of the number of critical points of a function on the manifold).[2]

We start with formulating the following simpler assertion. It was stated by Poincaré as a conjecture and proved by Birkhoff.

Theorem 1. *Suppose that a self-diffeomorphism of a circular annulus preserves area and moves the points of each of the boundary circles in the same direction and the points of different circles in opposite directions* (Fig. 1)*. Then this diffeomorphism has at least two fixed points.*

This assertion follows from a slightly more general theorem about fixed points of diffeomorphisms of the torus.

Theorem 2. *Let F be a diffeomorphism of the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ defined by $x \mapsto x + f(x)$ in the standard coordinate system. Suppose that F preserves area and "preserves the center of gravity," i.e., the mean value of the function f* (*considered as a function on the torus with standard metric*) *is zero. Then F has at least four fixed points.*

This is related to the fact that the sum of Betti numbers for the torus is equal to 4.

The first proof of this theorem was obtained by Y. Eliashberg. But nobody had verified this proof. A surely correct proof was published in 1983 by Conley and Zehnder, and this proof initiated a whole large theory – symplectic topology (developed by Chaperon, Laudenbach, Sikorav, Chekanov, Gromov, Floer, Hofer, Givental, and many other authors).[3] In recent months, there have been

[2] V. I. Arnold. On one topological property of globally canonical mappings of classical mechanics. In V. I. Arnold. *Selected Works–60* (Moscow: Fazis, 1997) [in Russian], pp. 81–86. (*Editor's note*)

[3] On symplectic topology see, e.g., V. I. Arnold. The first steps of symplectic topology. In V. I. Arnold. *Selected Works–60* (Moscow: Fazis, 1997) [in Russian], pp. 365–389. See also the references cited on p. XL of this book. (*Editor's note*)

communications that the initial conjectures (that the number of fixed points of an exact symplectomorphism is not smaller than the minimum number of critical points of a function on the manifold, at least for symplectomorphisms and generic functions) had been proved at last (by several independent groups in different countries).

2. Another example: the passage from $\mathbb{R}$ to $\mathbb{C}$. Using this example, it is, possibly, easier to explain the essence of the matter. We shall consider the passage from the real case to the complex one. There are real geometry and complex geometry. How can we pass from real geometry to complex geometry? For example, in real geometry, there is the notion of manifold with boundary, on which the notions of homology and homotopy are based. In general, the whole of topology essentially uses the notion of boundary.

We may ask: What becomes of the notion of boundary under complexification?

If we admit that all mathematics can be complexified, then, in particular, we must admit that various notions of mathematics can be complexified. Let us compose a table of transformations of various mathematical notions under complexification.

The complexification of the real numbers is, obviously, the complex numbers. Here the matter is very simple.

In the real case, there is Morse theory. Functions have critical points and critical values. Morse theory describes how level sets change when passing through critical values. What shall we obtain if we try to complexify Morse theory?

The complexification of real functions is holomorphic (complex analytic) functions. Their level sets have complex codimension 1, i.e., real codimension 2. In particular, they do not split the ambient manifold; the complement to a level set is by no means disconnected.

In the real case, the set of critical values of a function does split the real line. Therefore, generally, passing through a critical value affects the topology of a level set. For the complex analytic functions, this is not so. Their sets of critical values do not split the plane of the complex variable. Therefore, in the complex case, the level sets of a function that correspond to different noncritical values have the same topological structure. But in going around a critical value, a monodromy arises. This is a self-mapping of the set level (determined up to isotopy).

In the real case, the complement to a critical value consists of two components; thus, its homotopy group π_0 is $\mathbb{Z}_2$. In the complex case, the complement to a critical value is connected and has fundamental group $\mathbb{Z}$. Therefore, it is natural to regard π_1 as the complexification of π_0 and the group $\mathbb{Z}$ as the complexification of the group $\mathbb{Z}_2$.

Going further, we see that this approach turns out to be quite consistent. The complexification of the Morse surgeries (which refer to elements of the group π_0 of the set of noncritical values of a real function) is a monodromy (a representation of the group π_1 of the set of noncritical values of a complex function).

Monodromies are described by the Picard–Lefschetz theory, which is a theory of branching integrals.[4] In this sense, as the complexification of Morse theory we can regard the Picard–Lefschetz theory. As the complexification of a Morse surgery (attachment of a handle to a level set) we take the monodromy in a neighborhood of a nondegenerate critical point of a holomorphic function. This operation is the so-called Seifert transformation, that is, twisting a cycle on a level set. It consists in twisting a cylinder in such a way that one base of the cylinder remains fixed and the other base makes a full turn. In both cases, real and complex, the simplest operations correspond to singular points determined by sums of squares.

We can go even further. In the real theory, there are Stiefel–Whitney classes with values in $\mathbb{Z}_2$. Under complexification, they become Chern classes with values in $\mathbb{Z}$. Everything is consistent: the complexification of $\mathbb{Z}_2$ is indeed $\mathbb{Z}$.

The complexification of the projective line $\mathbb{R}\mathrm{P}^1 = S^1$ is the complex projective line $\mathbb{C}\mathrm{P}^1 = S^2$ (the Riemann sphere). Thus, the Riemann sphere is the complexification of the circle. It contains a circle (the equator). On this sphere, there is the theory of Fourier series defined on this circle and the theory of Laurent series which have two poles (at the poles of the sphere).

Let us find out what the complexification of the boundary of a real manifold is. First, we must algebraize the notion of boundary. A manifold with boundary is specified by an inequality of the form $f(x) \geqslant 0$. The correct complexification of this inequality is the equation $f(x) = y^2$. This equation specifies a hypersurface in the (x, y)-space, the standard projection of which on the x-space determines a double branched covering with branching along the boundary. Thus, the complexification of a manifold with boundary is a double covering with branching over the complex boundary.

This approach proved very fruitful. I invented the trick with a covering in 1970, when working on Hilbert's 16th problem about the arrangement of ovals of an algebraic curve of given degree n.[5] A polynomial of degree n in two variables determines a set of curves in the (real projective) plane. Hilbert's

[4] See V. A. Vasil'ev. *Branching integrals* (Moscow: MTsNMO, 2000) [in Russian]. (*Editor's note*)

[5] References to the literature on Hilbert's 16th problem are cited in D. Hilbert. *Selected Works* (Moscow: Factorial, 1998) [in Russian], vol. 2, p. 584. (*Editor's note*)