LONDON MATHEMATICAL SOCIETY STUDENT TEXTS

All the titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing, visit
http://publishing.cambridge.org/stm/mathematics/lmst

50. *A brief guide to algebraic number theory*, H. P. F. SWINNERTON-DYER
51. *Steps in commutative algebra, 2nd edn*, R. Y. SHARP
52. *Finite Markov chains and algorithmic applications*, O. HÄGGSTRÖM
53. *The prime number theorem*, G. J. O. JAMESON
54. *Topics in graph automorphisms and reconstruction*, G. DAVIDOFF, P. SARNAK & A. VALETTE
55. *Logic, induction and sets*, T. FORSTER
56. *Introduction to Banach algebras, operators and harmonic analysis*, H. G. DALES, P. AIENA, J. ESCHMEIER, K. LAURSEN & G. WILLIS
57. *Computational algebraic geometry*, H. SCHENCK
58. *Frobenius algebras and 2D topological quantum field theories*, J. KOCK

Singular Points of Plane Curves

C. T. C. Wall
University of Liverpool

CAMBRIDGE UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011–4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published 2004

Printed in the United Kingdom at the University Press, Cambridge

Typeface Computer Modern 10/13pt *System* LaTeX 2ε [AUTHOR]

A catalogue record for this book is available from the British Library

Library of Congress Cataloguing in Publication data

Wall, C. T. C. (Charles Terence Clegg)
Singular points of plane curves/C. T. C Wall.
p. cm. – (London Mathematical Society student texts; 63)
Includes bibliographical references and index.
ISBN 0 521 83904 1 (hardback) – ISBN 0 521 54774 1 (paperback)
1. Singularities (Mathematics) – Congresses. 2. Curves, Plane – Congress.
I. Title II. Series.

QA614.58.W35 2004
512′.22 – dc22 2004052121

ISBN 0 521 83904 1 hardback

ISBN 0 521 54774 1 paperback

Cambridge University Press
0521547741 - Singular Points of Plane Curves
C. T. C. Wall
Frontmatter
More information

Contents

Preface

The study of singular points of algebraic curves in the complex plane is a meeting point for many different areas of mathematics. The beginnings of the study go back to Newton. During the nineteenth and early twentieth century algebraic geometers working on plane curves developed methods which allowed them to deal with singular curves: see e.g. [168], [84], and several articles in volume III of the mathematical encyclopaedia published 1906–1914. A notable achievement was the resolution of singularities of such curves. In the late 1920s results in the then new area of topology were applied to the knots and links in the 3-sphere obtained by looking at the neighbourhood of such a singularity. There was a resurgence of interest about 1970 due to the interaction with newly developing ideas from singularity theory in higher dimensions, most importantly, the fibration theorem which Milnor had just discovered, in the context of functions of several complex variables. There has been continuous development since then, a particular point of interest being the application of Thurston's (circa 1980) decomposition theorems for 3-manifolds and for homeomorphisms of 2-manifolds.

The interaction between ideas from these different sources makes the study of curve singularities particularly fruitful and exciting. Equisingularity is an equivalence relation which admits characterisations from numerous differing points of view. The development of the ideas leading up to this is the leitmotif of the first half of this book. I thus emphasise the equivalence of different approaches, and feel that many results gain in clarity from appearing in an integrated account.

This book is based on an M.Sc. course given a number of times at the University of Liverpool. On the first such occasion (Autumn 1975) the course was given jointly by myself and two colleagues: Hugh Morton and Peter Newstead. It is a pleasure to acknowledge the insights derived from our collaboration on that occasion, and conversations subsequently.

The chapter on preliminaries is included to define a starting point: the topics mentioned here are not covered in detail. The core of the book is contained in Chapters 2–5, which lead to the equivalence of a range of conditions defining equisingularity. Here the level of exposition has been kept to that of the M.Sc. course (though I have added the decomposition theorem for general polar curves). We begin in Chapter 2 with a proof that a curve given by an equation may be represented by a particular type of parametrisation. Some foundational material on complex analysis is included to enable questions of convergence to be dealt with. It is then easy to proceed in Chapter 3 to a proof of resolution, and this in turn leads on naturally to a discussion of the invariants and configurations arising in the resolution process. The key concept in Chapter 4 is that of order of contact: this is developed to a flexible tool for giving the relation with intersection numbers and for answering questions that arise in the case of a curve with several branches. In Chapter 5 we begin the discussion of topology with a detailed geometrical picture of the knot, and proceed to calculate its Alexander polynomial, which suffices for the application to equisingularity. This fits well for students attending a parallel course on knot theory.

The later chapters are written at a more sophisticated level, and include introductions to a number of topics of recent research. The next two chapters deal with topics only briefly mentioned in the M.Sc. course. Chapter 6 contains proofs of Milnor's fibration theorems, first remarks about the Milnor fibre, and several calculations of Milnor numbers, emphasising the use of the Euler characteristic. Then we treat curves in the complex projective plane, with proofs of the general form of Plücker's theorems, Viro's proof of Klein's equation (using Euler characteristics of constructible functions), and an analysis of singularities of dual curves; and conclude with a survey of known results about curves whose singularities are maximal in some sense.

The next three chapters lead up to the calculation of the monodromy of the Milnor fibration. Chapter 8 introduces calculations and notation for later results in the form of several numerical invariants and their representation using exceptional cycles on a resolution tree. We include an introduction to the topological zeta function. Chapter 9 analyses the Thurston decomposition of the Milnor fibre and the JSJ decomposition of the link complement (following Le–Michel–Weber and Eisenbud–Neumann). Students attending a suitable parallel course, for example following [36], will find this an interesting application. A plumbing description of the neighbourhood of the resolution graph is given in sufficient detail for application to the study of the monodromy: it follows that the monodromy can be chosen to have no fixed points, and we obtain a direct relation of the Eggers tree and the resolution tree. We discuss how to calculate the E–N invariants, and give several necessary and sufficient conditions for the monodromy to have (pointwise) finite order. Chapter 10 opens with the definition of the Seifert matrix, and its interrelation with monodromy and intersection form, and proceeds to a detailed calculation of the monodromy map in homology, using the Thurston decomposition to define the weight filtration. It is shown how to classify Seifert forms over a field, and some of the invariants required for classification over $\mathbb{Q}$ are calculated.

In Chapter 11 we discuss ideals in relation to resolution, and a representation by exceptional cycles. There is a relation between ideals and clusters of infinitely near points, which takes the form of a Galois connection, in which a cluster is closed if and only if it satisfies the proximity inequalities and an ideal is closed if and only if it is valuatively closed. Enriques' unloading process is seen to be intimately related to the Zariski decomposition of cycles. There is a neat formula for the codimension of a closed ideal. The equivalence of valuative and integral closure is proved as an application of Noether's $Af + Bg$ theorem. We conclude with brief treatments of determinacy and of differential forms.

The later chapters can also be viewed as forming two parallel but interdependent developments; the geometry of the link complement and the Milnor fibration being studied in Chapters 6, 9 and 10; and the algebraic and combinatorial information being developed in Chapters 4, 8 and 11.

Each chapter is concluded by sections on 'Notes' and 'Exercises'. The notes include historical remarks, references – which we do not in general include in the main text – comments on related material (for example, characteristic p and the real as opposed to the complex case), and some references for further developments. The exercises include routine exercises on applying the results in the text to specific examples, and problems related to an alternative approach to a topic treated in the text.