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Karen M. Brucks and Henk Bruin

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KAREN M. BRUCKS

University of Wisconsin, Milwaukee

HENK BRUIN

University of Surrey



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Contents

<i>List of Figures</i>	<i>page</i> viii
<i>Preface</i>	xi
1 Topological Roots	1
1.1 Basics from Topology	1
1.2 Middle Third Cantor Set	9
2 Measure Theoretic Roots	12
2.1 Basics of Lebesgue Measure on \mathbb{R}	12
2.2 A Nonmeasurable Set	15
2.3 Lebesgue Measure of Cantor Sets	16
2.3.1 The Middle Third Cantor Set	16
2.3.2 Other Cantor Sets	17
2.4 Sets of Lebesgue Measure Zero	18
3 Beginning Symbolic and Topological Dynamics	19
3.1 Periodic Behavior	20
3.2 Nonwandering and ω -Limit Sets	24
3.3 Topological Conjugacy	33
3.4 Transitive Behavior	35
3.5 Recurrence	42
3.6 Shift Spaces	46
4 Beginning Measurable Dynamics	52
4.1 Preliminaries	52
4.2 Measurable Maps on I	53
4.3 Poincaré Recurrence	56
4.4 Ergodicity	58
4.4.1 Integration of Measurable Functions	60

4.4.2	Averaging Measurable Functions Along Orbits	62
4.4.3	A Connection to Topological Dynamics	65
5	A First Example: The 2^∞ Map	66
5.1	Logistic Family	66
5.2	A Bit of Combinatorics	68
5.3	Construction of the Cantor Set $\omega(c, g)$	68
5.4	Cantor Set and Adding Machines	70
5.5	A Toeplitz Sequence	73
6	Kneading Maps	74
6.1	Hofbauer Towers and Kneading Maps	74
6.2	First Uses of Kneading Maps	80
6.3	Shadowing	85
6.4	Examples of Kneading Maps	87
7	Some Number Theory	92
7.1	Farey Tree	92
7.2	Continued Fractions	96
7.3	Continued Fractions and the Farey Tree	98
8	Circle Maps	101
8.1	Circle Homeomorphisms	101
8.2	Degree One Circle Maps	105
8.3	Irrational Rotations and Return Maps	111
8.4	Cantor Thread	114
9	Topological Entropy	117
9.1	Basic Properties of Topological Entropy	118
9.2	Entropy of Subshifts	123
9.3	Lapnumbers and Markov Extensions	129
9.4	Lapnumbers and Entropy	136
9.5	Semiconjugacy to a Piecewise Linear Map	140
9.6	The Monotonicity Problem	144
10	Symmetric Tent Maps	147
10.1	Preliminary Combinatorics	148
10.2	ω -Limit Sets	153
10.3	Phase Portrait	157
10.4	Measure Results	165
10.5	Slow Recurrence and the CE Condition	168
10.6	Attractors	172
10.7	Combinatorics and Renormalization	174
11	Unimodal Maps and Rigid Rotations	178
11.1	Adding Machines in Unimodal Maps	178
11.2	Rigid Rotations in Unimodal Maps – I	184
11.3	Rigid Rotations in Unimodal Maps – II	186

Cambridge University Press
 978-0-521-54766-6 - Topics from One-Dimensional Dynamics
 Karen M. Brucks and Henk Bruin
 Frontmatter
[More information](#)

<i>CONTENTS</i>	vii
12 β-Transformations, Unimodal Maps, and Circle Maps	193
12.1 β -Transformations and β -Expansions	193
12.2 Flip-Half-of-the-Graph Trick	195
12.3 A Relation Between Unimodal Maps and Circle Maps	197
12.4 Comparing β -Transformations and Tent Maps	203
12.5 Ledrappier's Example	208
12.6 Maps with Slope < 2	211
13 Homeomorphic Restrictions in the Unimodal Setting	216
13.1 First Observations	218
13.2 A 2^∞ Trapezoidal Map	220
13.3 The Adding Machine (Ω, \mathbb{P})	225
13.4 The Case $Q(k) \rightarrow \infty$	238
14 Complex Quadratic Dynamics	250
14.1 Julia Sets and External Rays	251
14.2 The Mandelbrot Set	262
14.3 Itineraries and Hubbard Trees	266
<i>Bibliography</i>	279
<i>Index</i>	292

List of Figures

1	Trail guide	<i>page</i> xii
1.1	Construction of the Middle Third Cantor set	10
3.1	Iterates of $g_a(x) = ax(1 - x)$ for $a \approx 3.8318740552833155684$	21
3.2	Graph of T_a for $a = 1.5$	21
3.3	Two period 5 orbits and associated permutations	22
3.4	Example of repelling fixed point	24
3.5	Open sets G_1 and G_2	27
3.6	Graphs of T_2 , T_2^2 , and T_2^3	28
3.7	Placement of z_2	29
3.8	Graph of g	33
3.9	Topological conjugacy	34
3.10	Once renormalizable map g_a	40
3.11	Construction of $r_n(x)$	45
3.12	Some transition graphs with transition matrices	49
3.13	Transition matrix B for $X_{\mathcal{F}=\{11\}}$	50
5.1	Construction of $\omega(c, \mathbf{g})$	70
5.2	Commuting diagram	72
6.1	Core of f	75
6.2	Central branches	76
6.3	Hofbauer tower for Fibonacci combinatorics	77
6.4	Geometry of admissibility condition	79
6.5	Choice of B and construction of V	82
6.6	Construction of $\{c_{-j}\}_{j \geq m}$	83
6.7	View of D_n and $D_{n-S_{k-1}}$	84
6.8	Even and odd returns	86
6.9	Property HK-1	88
7.1	Farey tree	93
8.1	Rigid rotation R_α	103
8.2	Maps F , F_l , and F_u	108

LIST OF FIGURES

ix

8.3	First return map $\mathcal{R}(R_\rho)$ rescaled to $R_{\rho'}$ for $0 < \rho < 1/2$	112
8.4	First return map $\mathcal{R}(R_\rho)$ rescaled to $R_{\rho'}$ for $1/2 < \rho < 1$	113
8.5	Construction of a homeomorphism h for given γ	116
9.1	Graph of T^3 on the core $[c_2, c_1]$	128
9.2	Graph G_A	128
9.3	An interval map and some vertices of its Markov extension.	131
9.4	Left: The Markov extension for the Fibonacci map, with the Hofbauer tower in thick lines. Right: The transition matrix of the Hofbauer tower	133
9.5	Transition graph for the Feigenbaum map	134
9.6	Piecewise linear maps semiconjugate to a fixed map	144
9.7	Barn map f_A	145
10.1	Coding of laps	150
10.2	Piece of $T_a^{S_k^{-1}}$	152
10.3	ω -limit sets for T_a	154
10.4	Phase portrait: ξ_1 to ξ_4	157
10.5	Prefixed parameters	160
10.6	Plotting $\xi_n(a)$ at cutting time $n = S_k$	161
10.7	Lap of T_a^{n-1} containing c_1	163
10.8	Graphs of ξ_n , r_n , and s_n	164
10.9	Graphical consequence of critically finite	171
10.10	$c \in \text{Comp}(f^j(x), f^{-(n_i-j)}(B(f^{n_i}(x), \delta)))$ for $j = n_i - l$ and $n_i - m$	172
10.11	Turn at z after the graph has left and reentered the δ -band	172
10.12	First three iterates of T_a	175
10.13	First two iterates of T_b	175
11.1	$x = \pi(e) = \bigcap_{i \geq 0} D_{\omega(q_i)}$	183
11.2	Case $\rho < 1/2$ (not to scale)	189
11.3	Case $\rho > 1/2$ (not to scale)	191
12.1	Graph of f_β for some $\beta \in (2, 3)$	194
12.2	Graphs of τ_β and φ_β	196
12.3	Graphs of φ_g and $\bar{\varphi}_g$ with interval D	199
12.4	Graph of h_1 and its compositions	205
12.5	Graph of h_2 and its compositions	206
12.6	Graph of h_3 and its compositions	206
12.7	A Ledrappier three-dot pattern with \lrcorner -shaped patterns	208
12.8	Covers with \lrcorner -shaped patterns	209
12.9	A Ledrappier three-dot pattern for $e = 1111\dots$	210
12.10	Graphs of T_α and f_α with Markov partitions	212
13.1	Commuting diagram	218
13.2	Trapezoidal map $g_{(a,b,d)}$	220
13.3	Symmetric trapezoidal map g_e	221
13.4	Graph of g	222
13.5	Geometry of Condition A	238
13.6	Placement via kneading map	241
13.7	Nesting for $x = \pi(\omega)$	245
13.8	$c_{S_{q_0}} \in D_{\omega(q_1)}$	248
14.1	Stereographic projection	252
14.2	Julia set J_0 and external rays	254

14.3	Julia set J_{-2} and external rays	255
14.4	Julia set J_i	256
14.5	Choice of component U_0	259
14.6	Julia set $J_{i+0.5}$ with disks D , D_0 , and D_1	261
14.7	Mandelbrot set	262
14.8	Examples of trees and nontrees	268
14.9	Two configurations of the spine and its image	269
14.10	Abstract Hubbard trees for $1 \rightarrow 3 \rightarrow 5 \rightarrow 7 \rightarrow \dots$	271
14.11	Embeddings of branchpoints in the plane	275
14.12	The abstract Hubbard tree for $1 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 6$	278

Preface

One-dimensional dynamics owns many deep results and avenues of active mathematical research. Numerous inroads to this research exist for the advanced undergraduate or beginning graduate student. It is precisely these students whom we target. Several glimpses into one-dimensional dynamics are provided with the hope that the results presented illuminate the beauty and excitement of the field. Many topics covered appear nowhere else in “textbook format,” some are mini new research topics in themselves, and for nearly all topics we try to provide novel connections with other research areas both inside and outside the text. Among these topics are kneading theory and Hofbauer towers; detailed structure of ω -limit sets; topological entropy; lapnumbers and Markov extensions; the 2^∞ map (Feigenbaum-Coulet-Tresser), interplay amongst continued fractions, adding machines, circle maps, and unimodal maps; irrational rotations as factors of unimodal maps; connections between β -transformations and unimodal maps; Ledrappier’s three-dot example; and itineraries for complex quadratic maps and Hubbard trees. The flavor is largely combinatoric, symbolic, and topological. The material presented is *not* meant to be approached in a linear fashion. Rather, we strongly encourage readers to pick and choose topics of interest. Trail routes (other than $n \mapsto n + 1$) are indicated in Figure 1; more explicit information is provided at the beginning of each chapter. Suggested uses for the text include: dynamics courses, master theses, reading courses, research experiences for undergraduates (REUs), seminars, senior projects, and summer courses.

As mentioned, the topics covered are *not* the typical topics seen in undergraduate/graduate dynamics texts. Rather, the material is a filtering from the research literature of currently active topics that can be made accessible to the targeted student audience. Frequent references to the literature are made to enable the reader to further investigate the topic under study. Occasionally new or simplified proofs for existing results are given. The level of “mathematical sophistication” required by the reader varies between chapters. The text is also designed to provide the reader with a “critical mass” of results and tools, allowing for further investigation into the field.

Although this text can be used in a “lecture format” course, we have purposely employed a style such that a more active role can be played by students. Exercises are not reserved for the end of a section but rather are sprinkled throughout a section. The student is encouraged to complete exercises as s/he progresses through the material. More challenging exercises are tagged with a \diamond . For some exercises, having referenced literature at hand is suggested and such exercises are tagged with a \clubsuit . The Java Applets, www.uwm.edu/~kmbrucks/Dyntext.html, assist with the exploration of many topics.

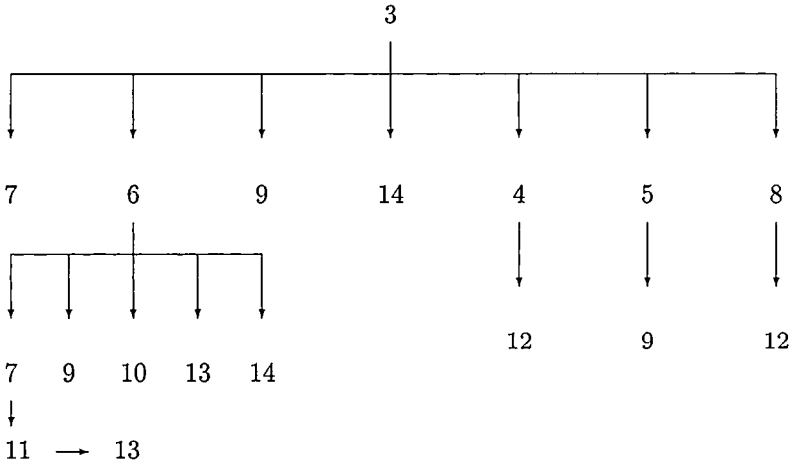


Figure 1: Trail guide

Historically, it was in the 1970s that dynamical systems had a “surge” of activity (which continues today). This surge was due largely to the accessibility of computers and the multitude of results and questions achieved with the aid of computers. However, a genealogical tree of dynamics would include G. D. Birkhoff (1884–1944), M. L. Cartwright (1900–1998), P. Fatou (1878–1929), G. Julia (1893–1978), and H. Poincaré (1854–1912). Historical discussions can be found in [4, 5, 108, 115].

Research in one-dimensional dynamics was ignited in the 1960s and 1970s by American, French, and Ukrainian mathematicians [58, 68, 69, 107, 116, 122, 155, 160]. Metric properties of the Coulet-Tresser (or Feigenbaum) map (see Chapter 5) were studied in the late 1970s [58, 68, 69] and somewhat earlier (and more narrowly) in [127]. The famous theorem of Sharkovsky (see Theorem 3.1.5) appeared in a Ukrainian journal in 1964 [155]. Work of Stefan [160] in 1977 helped bring Sharkovsky’s theorem to the west. Around this same time, Li and Yorke published their famous paper “Period

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Frontmatter

[More information](#)

xiii

three implies chaos" [107]. Published works on the combinatorics of one-dimensional dynamics first appeared in the early 1970s [116, 122]. From these roots, one-dimensional dynamics has flourished into an active and exciting area of mathematical research.

Chapters 1 and 2 provide background material from topology and measure theory, respectively. Rather than begin with these chapters, we suggest the reader refer to these chapters as needed. These chapters are purposely succinct and provide multiple references for the reader looking for more detail/discussion. Lebesgue measure is the only measure used, and it appears infrequently. Hence the reader unfamiliar with measure theory can still access most of the text.

Chapter 3 provides an introduction to symbolic and topological dynamics. This material provides a dynamical foundation for the topics covered in later chapters. We suggest that the reader who is unfamiliar with this material work through the first four sections and move on, returning for the remaining material (recurrence and shift spaces) as needed. A beginning discussion on measurable dynamics is given in Chapter 4. Pieces of this material are referenced in Chapter 12; however, the material is not required for Chapter 12.

The dynamics of a 2^∞ map are investigated in Chapter 5. In particular, the dynamics of this map are shown to be that of the dyadic adding machine. The discussion (in Chapter 5) is accessible to an undergraduate student with minimal experience doing proofs. Generalized versions of the dyadic (triadic, etc.) adding machine are investigated in Chapters 11 and 13.

Chapters 6 and 7 provide combinatoric, number theoretic, and symbolic machinery to investigate one-dimensional dynamical systems. Chapters 8 to 14 target the student with a higher level of mathematical sophistication. Any one of these chapters (8–14) could serve as a component of a dynamics course or as material for a reading course, masters thesis, seminar, summer course, or term project. Of course, this text is not a comprehensive work on one-dimensional dynamics. Other texts in dynamics include [4, 5, 18, 53, 56, 64, 74, 96, 108, 115, 139, 156, 162, 168].

We thank Kurt Cogswell, Beverly Diamond, Jane Hawkins and her students, Chris Sears, and Lubo Snoha for their careful reading of the text; Jane Hawkins for contributing Chapters 2 and 4; Karsten Keller for the figure files, Peter Raith for e-mail discussions on Hofbauer towers; and Christopher Sears for the Java Applets.

This text grew out of courses taught (1997, 1999, 2001) at the Carleton Summer Mathematics Program (SMP) for undergraduate women in mathematics. Each summer, the SMP draws 18 talented undergraduate women

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Frontmatter
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xiv

PREFACE

interested in mathematics to Carleton College for four weeks of intensive mathematics study. We applaud Deanna Haunsperger and Steven Kennedy for running a program that so positively impacts all involved.

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