Chapter 1

Topological Roots

This chapter provides some basics from topology. Our style is purposely succinct as our objective is to collect, for the reader to reference as s/he works through the text, the results from topology needed for the text. The reader with minimal topological experience may want to keep an introductory topology text at hand, for example, [126, 170]. We suggest that others reference the chapter as needed.

Some notation: $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, $\mathbb{R}$ denotes the real numbers, and $\mathbb{Q}$ denotes the rationals. For ease of notation, $\langle a; b\rangle$ denotes the closed (unless otherwise stated) interval $[a, b]$ when $a < b$ and denotes $[b, a]$ when $b < a$. Let $\#A$ denote the cardinality of the set A and A^c the complement of A. For $A \subset X$ we also use $X \setminus A$ to denote the complement of A.

1.1 Basics from Topology

Definition 1.1.1. A *metric space* is an ordered pair (E, ρ) consisting of a set E along with a function $\rho : E \times E \to \mathbb{R}$ satisfying the following for $x, y, z \in E$:

1. $\rho(x, y) \geq 0$;
2. $\rho(x, x) = 0$ and $\rho(x, y) = 0$ implies $x = y$;
3. $\rho(x, y) = \rho(y, x)$;
4. $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$ (*triangle inequality*).

The function ρ is called a *metric* on E. If only conditions 1, 3, and 4 hold, we call ρ a *semimetric.*

Example 1.1.2. *The real line* $\mathbb{R}$ *with the distance function* $\rho(x, y) = |x-y|$ *is a metric space.*

Example 1.1.3. *The Euclidean space* $\mathbb{R}^n$ *with the distance function*

$$\rho(\langle x_1, \ldots, x_n\rangle, \langle y_1, \ldots, y_n\rangle) = \sqrt{\sum_{k=1}^{n}(x_k - y_k)^2}$$

is a metric space.

Example 1.1.4. *The Euclidean space* $\mathbb{R}^2$ *with the distance function*

$$\rho(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$$

is a metric space.

Example 1.1.5. *Set*

$$E = \{0, 1\}^{\mathbb{N}} = \{x_1, x_2, \ldots \mid x_i \in \{0, 1\} \text{ for all } i\}.$$

Thus, E consists of all one-sided infinite strings of 0*'s and* 1*'s. Let* $x = \{x_i\}_{i\geq 1}$ *and* $y = \{y_i\}_{i\geq 1}$ *be elements from* E *and set:*

$$\rho(x, y) = \sum_{i\geq 1} \frac{|x_i - y_i|}{2^i}.$$

Then (E, ρ) *is a metric space.*

Definition 1.1.6. Let (E, ρ) be a metric space and $x \in E$. For $\epsilon > 0$ we define the ϵ-*ball* about x as:

$$B(x, \epsilon) = \{y \in E \mid \rho(x, y) < \epsilon\}.$$

For subsets $S, T \subset E$, the distance between S and T is defined as:

$$\rho(S, T) = \inf\{\rho(x, y) \mid x \in S, \ y \in T\}.$$

The ϵ-ball about the set S is given as:

$$B(S, \epsilon) = \{y \in E \mid \rho(S, y) < \epsilon\}.$$

Definition 1.1.7. Let (E, ρ) be a metric space and $M \subset E$. We call M *open* provided that, for each $x \in M$, there is an ϵ-ball containing x and contained in M. A set is *closed* provided it is the complement of an open set.

Exercise 1.1.8. *Prove each of the following.*

1. *Any union of open sets is open.*
2. *Any finite intersection of open sets is open.*
3. *Construct an infinite collection of open sets whose intersection is NOT open.*

What happens if you replace "open" with "closed"? **HINT:** *For item (3), let* $U_n = (-\frac{1}{n}, \frac{1}{n})$ *for* $n \geq 1$. *Each* U_n *is open and yet* $\cap_n U_n = \{0\}$. *When considering closed sets, recall the following equalities from set theory:*

$$E \setminus (\cap_{\lambda \in \Lambda} U_\lambda) = \cup_{\lambda \in \Lambda} (E \setminus U_\lambda) \quad \text{and} \quad E \setminus (\cup_{\lambda \in \Lambda} U_\lambda) = \cap_{\lambda \in \Lambda} (E \setminus U_\lambda).$$

Definition 1.1.9. Let (E, ρ) be a metric space and $M \subset E$. The *interior* of M, denoted M°, is defined as $M^\circ = \cup\{U \subset E \mid U \text{ is open and } U \subset M\}$.

Definition 1.1.10. Let (E, ρ) be a metric space and $M \subset E$. A point $x \in E$ is said to be an *accumulation point of the set* M provided that, for every open set $U \ni x$, we have $M \cap (U \setminus \{x\}) \neq \emptyset$.

Definition 1.1.11. Let (E, ρ) be a metric space, $M \subset E$, and M' be the set of accumulation points of M. The *closure* of the set M, denoted $\overline{M}$, is defined to be $M \cup M'$.

Exercise 1.1.12. *Let* (E, ρ) *be a metric space,* $M \subset E$, *and* M' *be the set of accumulation points of* M. *Prove* M' *is closed.*

Definition 1.1.13. Let (E, ρ) be a metric space and $M \subset E$. The *boundary* of M, denoted ∂M, is defined as $\partial M = \overline{M} \cap \overline{E \setminus M}$.

Definition 1.1.14. Let (E, ρ) be a metric space. A sequence $\{x_n\}_{n \geq 1}$ in E *converges to* x in E provided $\lim_{n \to \infty} \rho(x_n, x) = 0$. If $\{x_n\}$ converges to x, we write $\lim_{n \to \infty} x_n = x$ or, more shortly, $x_n \to x$.

Exercise 1.1.15. *Let* (E, ρ) *be a metric space and suppose that* $\lim_{n \to \infty} x_n = x$ $(x_n \to x)$ *and* $\lim_{n \to \infty} x_n = y$ $(x_n \to y)$. *Show* $x = y$.

Exercise 1.1.16. *We say a sequence* $\{x_n\}_{n \geq 1}$ *of real numbers is* monotone increasing (monotone decreasing) *provided* $x_i \leq x_{i+1}$ *for all* $i \geq 1$ *(provided* $x_i \geq x_{i+1}$ *for all* $i \geq 1$*). We call* $\{x_n\}$ *monotone if it is either monotone increasing or decreasing. Prove that every monotone bounded sequence of real numbers converges.* **HINT:** *Assume the sequence is monotone increasing. Let* L *be the least upper bound of* $\{x_n\}_{n \geq 1}$. *Show* $\lim_{n \to \infty} x_n = L$.

Exercise 1.1.17. *Prove that every sequence of real numbers contains a monotone subsequence. Conclude (using Exercise 1.1.16) that every bounded sequence of real numbers has a convergent subsequence.* **HINT:** *Call* x_m *a* turn-point *provided* $x_n \le x_m$ *for all* $n > m$. *If there exist* $m_1 < m_2 < \dots$ *such that each* x_{m_i} *is a turn-point, then* $x_{m_1} \ge x_{m_2} \ge \dots$ *is a monotone subsequence. Show that if there are only finitely many (or no) turn-points, then one may choose* $x_{n_1} < x_{n_2} < \dots$.

Definition 1.1.18. Let (E_1, ρ_1) and (E_2, ρ_2) be metric spaces. A function $f : E_1 \to E_2$ is *continuous* at $x \in E_1$ provided that, for each $\epsilon > 0$, there is some $\delta > 0$ such that $\rho_2(f(x), f(y)) < \epsilon$ whenever $\rho_1(x, y) < \delta$.

Exercise 1.1.19. *Let* (E_1, ρ_1) *and* (E_2, ρ_2) *be metric spaces and* $f : E_1 \to E_2$. *Prove that the map* f *is continuous at* $x \in E_1$ *iff, for each open set* $V \subset E_2$ *containing* $f(x)$*, there is an open set* $U \subset E_1$ *containing* x *such that* $f(U) \subset V$. *Conclude that if* f *is continuous for all* x *in* E_1 *and* $W \subset E_2$ *is open (in* E_2*), then* $f^{-1}(W)$ *is open (in* E_1*).*

Exercise 1.1.20. *Let* $(E_1, \rho_1), (E_2, \rho_2)$ *be metric spaces and* $f : E_1 \to E_2$. *Prove that the map* f *is continuous if, whenever* $x_n \to x$ *in* E_1*, then* $f(x_n) \to f(x)$ *in* E_2.

Definition 1.1.21. Let (E, ρ) be a metric space and $M \subset E$. We say M is *nowhere dense in* E provided that $\overline{M}$ contains no nonempty open set.

The integers $\mathbb{Z}$ in $\mathbb{R}$ are nowhere dense in $\mathbb{R}$. The rationals $\mathbb{Q} \in [0, 1]$ are *not* nowhere dense in $[0, 1]$. We will soon get nontrivial examples of nowhere dense sets, namely, Cantor sets. First, some definitions.

Definition 1.1.22. Let (E, ρ) be a metric space and $M \subset E$. We say M is *perfect* provided M is closed and each point of M is an accumulation point of M.

Definition 1.1.23. Let (E, ρ) be a metric space. We call E *disconnected* provided there are disjoint nonempty open sets H and K in E such that $E = H \cup K$. If no such disconnection exists, then we call E *connected.*

Exercise 1.1.24. *Let* (E_1, ρ_1) *and* (E_2, ρ_2) *be metric spaces and* $f : E_1 \to E_2$ *a continuous onto map. Prove that if* E_1 *is connected, then* E_2 *is also connected.* **HINT:** *Suppose to the contrary that* E_2 *is not connected. Then there are nonempty open sets* H *and* K *in* E_2 *such that* $E_2 = H \cup K$. *However, then* E_1 *would be disconnected by* $f^{-1}(H)$ *and* $f^{-1}(K)$*, a contradiction.*

Definition 1.1.25. Let (E, ρ) be a metric space. We call E *totally disconnected* provided the only nonempty connected subsets of E are the one-point sets.

Notice that $M \subset \mathbb{R}$ is totally disconnected iff M contains no nonempty open set. For example, the rationals $\mathbb{Q}$ are totally disconnected.

Definition 1.1.26. A metric space E is *compact* if every sequence in E has a convergent subsequence in E (the limit of the convergent subsequence is, of course, in E).

Exercise 1.1.27. *Let (E, ρ) be a compact metric space and $f : E \to E$ continuous. Prove that $f(E)$ is compact.* **HINT:** *Recall Exercise 1.1.20.*

Exercise 1.1.28. *Let (E, ρ) be a compact metric space. Prove that $M \subset E$ is compact if and only if M is closed.* **HINT:** *Assume M is compact. If M is not closed, then $E \setminus M$ is not open, and hence one can choose $x \in E \setminus M$ and $x_n \in M \cap B(x, \frac{1}{n})$ for each $n \in \mathbb{N}$. As M is compact, a subsequence of $\{x_n\}$ converges to some $y \in M$, contradicting $\lim_{n\to\infty} x_n = x \notin M$. Next assume M is closed and that $\{x_n\}$ is a sequence in M. We need to find a convergent (in M) subsequence. As E is compact, we obtain a convergent (in E) subsequence $\{x_{n_j}\}$; say $\lim_{j\to\infty} x_{n_j} = y \in E$. We want to show that $y \in M$. If not, then (since M is closed) there is $B(y, \epsilon) \subset E \setminus M$; this contradicts $\{x_{n_j}\}$ converging to y with each $x_{n_j} \in M$.*

The interested reader should see a standard topology text for proofs of Theorems 1.1.29 and 1.1.34 [126, 170]. We prove Theorem 1.1.29 for the case $n = 1$.

Theorem 1.1.29 (Bolzano-Weierstrass [170]). *A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.*

Proof. We do only the case $n = 1$.

Assume $E \subset \mathbb{R}$ is closed and bounded, and let $\{x_n\}_{n\geq 1}$ be a sequence from E. By Exercise 1.1.17, we may choose a monotone subsequence $\{x_{n_k}\}$. As E is bounded, so is the sequence $\{x_n\}$ and the subsequence $\{x_{n_k}\}$. By Exercise 1.1.16, the subsequence converges. The limit of the subsequence is in E as E is closed. Hence, E is compact.

Assume $E \subset \mathbb{R}$ is compact. If E were unbounded, then one could construct a sequence $\{x_n\}_{n\geq 1}$ such that $x_n \to \infty$ (or $x_n \to -\infty$), contradicting E being compact (as no subsequence converges). Thus, E is bounded. If E is not closed, then we may choose an accumulation point z of E in $\mathbb{R}\setminus E$. As z is an accumulation point of E, we may choose a sequence $\{x_n\}$ from E converging to z. This sequence has no subsequence converging to a point in E, contradicting E being compact. Thus, E is indeed closed and bounded. ☐

Exercise 1.1.30. *Let (E, ρ) be a compact metric space and $M, N \subset E$ with $M \cap N = \emptyset$ and M, N closed. Show that $\rho(M, N) > 0$. Is this true if E is not compact?* **HINT:** *For the case E not compact, let $E = \mathbb{R}^2$, $M = \mathbb{R}$, and N be the graph of $\frac{1}{x}$.*

Definition 1.1.31. Let (E, ρ) be a metric space and $f : E \to E$ continuous. We say that f is *uniformly continuous* provided that, for every $\epsilon > 0$, there exists $\delta > 0$ such that $x, y \in E$ and $|x - y| < \delta$ imply $|f(x) - f(y)| < \epsilon$.

Exercise 1.1.32. *Let (E, ρ) be a compact metric space and $f : E \to E$ continuous. Prove that f is uniformly continuous.* **HINT:** *Suppose to the contrary that f is not uniformly continuous. Choose $\epsilon > 0$ and for each $n \in \mathbb{N}$ choose p_n, q_n such that $\rho(p_n, q_n) < \frac{1}{n}$ and $\rho(f(p_n), f(q_n)) \geq \epsilon$. As E is compact, choose $\{n_k\}$ and z such that $\lim_{k\to\infty} p_{n_k} = z$. Then $\lim_{k\to\infty} q_{n_k} = z$ also. Hence, $\lim_{k\to\infty} f(p_{n_k}) = \lim_{k\to\infty} f(q_{n_k}) = f(z)$; this contradicts $\rho(f(p_{n_k}), f(q_{n_k})) \geq \epsilon$ for all k.*

Exercise 1.1.33. *Let $f : (0, 1] \to [1, \infty)$ be given by $f(x) = 1/x$. Is f uniformly continuous?*

Theorem 1.1.34 (Heine-Borel [170]). *A subset M of a metric space (E, ρ) is compact iff, whenever M is contained in the union of a collection of open sets of E, then M is also contained in the union of a finite subcollection of these sets.*

Lemma 1.1.35 (Lebesgue covering lemma [170]). *Let (E, ρ) be a compact metric space and $\{U_1, \dots, U_n\}$ be a finite open cover of E, that is, $\cup_{i=1}^{i=n} U_i = E$ and each U_i is open. There exists $\delta > 0$ such that, if A is any subset of E of diameter less than δ, then $A \subset U_i$ for some i.*

Proof. Suppose not. For each $n \in \mathbb{N}$, let A_n be a set of diameter less than $\frac{1}{n}$ such that $A_n \not\subseteq U_i$ for any i. Let $x_n \in A_n$ for each n. As E is compact, there exists $\{n_k\}$ and $x \in E$ with $\lim_{k\to\infty} x_{n_k} = x$. Choose i such that $x \in U_i$ and $\delta > 0$ such that $B(x, \delta) \subset U_i$. Choose k such that $\frac{1}{n_k} < \frac{\delta}{2}$ and choose $l > k$ such that $x_{n_l} \in B(x, \frac{\delta}{2})$. Then $A_{n_l} \subset B(x, \delta) \subset U_i$, a contradiction. □

Exercise 1.1.36. *Let (E, ρ) be a compact metric space and $f : E \to E$ a continuous map. Prove, directly, that f is uniformly continuous.* **HINT:** *Fix $\epsilon > 0$. For each $x \in E$, choose $\delta_x > 0$ such that $\rho(x, y) < \delta_x$ implies $\rho(f(x), f(y)) < \frac{\epsilon}{2}$. Use Theorem 1.1.34 to choose $x_1, x_2, \dots, x_n \in E$ such that $\cup_{i=1}^{i=n} B(x_i, \delta_{x_i}) \supset E$. Use Lemma 1.1.35 to choose $\delta > 0$ such that $\rho(x, y) < \delta$ implies $x, y \in B(x_i, \delta_{x_i})$ for some i. Conclude that $\rho(x, y) < \delta \Rightarrow \rho(f(x), f(y)) < \epsilon$.*

Definition 1.1.37. Let (E_1, ρ_1) and (E_2, ρ_2) be compact metric spaces and $f : E_1 \to E_2$. We call f a *homeomorphism* provided f is one-to-one and both f and f^{-1} are continuous (the domain of f^{-1} being $f(E_1)$). If, additionally, f is onto, we call f an *onto homeomorphism*.

Exercise 1.1.38. ♣ *[170, Theorem 17.14] Let $h : X \to X$ be continuous, onto, and one-to-one. Assume X is compact. Prove h^{-1} is continuous and hence is a homeomorphism.*

Definition 1.1.39. Let (E, ρ) be a metric space. We call E a *Cantor set* provided E is compact, totally disconnected, and perfect.

In Section 1.2 we construct the *Middle Third Cantor set.* This set is a standard first example of a Cantor set. Any two Cantor sets are homeomorphic [170]. However, we have the stronger result that Cantor sets are *homogeneous*; see Exercise 1.1.40. This property is used in the proof of Proposition 1.1.52.

Exercise 1.1.40. ♣ *[170, page 218] Let K_1 and K_2 be Cantor sets. Fix $x_1 \in K_1$ and $x_2 \in K_2$. Prove there exists a homeomorphism $h : K_1 \to K_2$ with $h(x_1) = x_2$. Due to this property, we say Cantor sets are* homogeneous.

Definition 1.1.41. We call a set M *countable* if one of the following hold.

- There is a bijection (one-to-one and onto map) between M and $\mathbb{N}$.
- The set M is finite or empty.

Exercise 1.1.42. *Prove*

1. *$\mathbb{Q}$ is countable.*
2. *$\mathbb{R}$ is not countable.*

Exercise 1.1.43. *Let M be a countable set. Prove that $M^2 = \{(m_1, m_2) \mid m_i \in M \text{ for each } i\}$ is countable. More generally, show that, for each $n \in \mathbb{N}$, the set M^n is countable. Is $M^{\mathbb{N}}$ countable?*

Remark 1.1.44. Let (E, ρ) be a compact metric space and suppose $M \subset E$ is closed (and therefore compact) and perfect. Then M is not a countable set and hence is uncountable [170, Exercise 30B-2]. For further discussion see [170, Chapter 30].

Definition 1.1.45. Let (E, ρ) be a metric space. We say $D \subset E$ is *dense in E* provided $\overline{D} = E$. We call E *separable* provided E contains a countable dense subset.

The rationals $\mathbb{Q}$ are dense in $\mathbb{R}$ and hence $(\mathbb{R}, \rho)$ (ρ given in Example 1.1.3) is a separable metric space.

Exercise 1.1.46. *Let (E, ρ) be a separable metric space. Show there exists a countable collection of open sets, $\mathcal{U}$, with the property that, given $x \in E$ and $W \ni x$ open in E, there exists $U \in \mathcal{U}$ with $x \in U \subset W$.* **HINT:** *Let $\{e_1, e_2, e_3, \dots\}$ be a countable dense subset from E. Set $\mathcal{U} = \{B(e_i, \frac{1}{m}) \mid i, m \in \mathbb{N}\}$.*

Exercise 1.1.47. $\diamondsuit$ *Prove that every compact metric space is separable.*

Definition 1.1.48. Let (E, ρ) be a metric space, $M \subset E$, and $x \in M$. We call x a *condensation point of M* provided each open $U \ni x$ contains uncountably many points of M.

Lemma 1.1.49. *Let (E, ρ) be a separable metric space and $M \subset E$. Then M can be expressed as $M = F \cup H$, where H is countable and every point of F is a condensation point of M.*

Proof. The result is clear if M is countable (simply take $H = M$ and $F = \emptyset$). Hence, assume M is uncountable.

Let $\mathcal{U}$ be as in Exercise 1.1.46, and set

$$H = \{x \in M \mid x \text{ is not a condensation point of } M\}.$$

We show H is countable. For each $x \in H$, there is $U_x \in \mathcal{U}$ such that $x \in U_x$ and $U_x \cap M$ is countable. As $H \subset M$, we have that $U_x \cap H$ is countable. Lastly, as $\mathcal{U}$ is countable, we have H countable. Set $F = M \setminus H$. □

Exercise 1.1.50. *Let (E, ρ) be a compact metric space and $M \subset E$. Write $M = F \cup H$ as in Lemma 1.1.49. If $F \neq \emptyset$, prove F is perfect.*

Exercise 1.1.51. *Let $M \subset [0, 1]$ be closed, uncountable, and contain no nonempty open set. Then one can express M as $M = F \cup H$, where F is a Cantor set and H is countable (it can be that $H = \emptyset$).*

Proposition 1.1.52. *Let (E, ρ) be a compact metric space, and let $M \subset E$ be closed, uncountable, and totally disconnected. Then M can be uniquely expressed as the disjoint union of a Cantor set and a countable set.*

Proof. Express $M = F \cup H$ as in Lemma 1.1.49. Thus, every point of F is a condensation point of M, F is a Cantor set (recall Exercise 1.1.50), and H consists of those points of M that are not condensation points of M. Note that $F \cap H = \emptyset$. Suppose $M = F_1 \cup H_1$ with F_1 a Cantor set, H_1 countable, and $F_1 \cap H_1 = \emptyset$.

We show $H_1 \subset H$. Suppose not, that is, suppose $z \in H_1 \cap F$. As F_1 is closed and $z \notin F_1$, we may choose open $U \ni z$ such that $U \subset H_1$. Thus U is a countable set. However, $z \in F$ and U open implies $U \cap F$ is uncountable. This contradiction shows that indeed $H_1 \subset H$.

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Lastly, we show $H_1 = H$. If H_1 is a proper subset of H, then $F_1 \cap H \neq \emptyset$. Let $x \in F \cap F_1$ and $y \in F_1 \cap H$. By Exercise 1.1.40, we may choose a homeomorphism $h : F \to F_1$ with $h(x) = y$. Since $y \in H$, it is the case that y is a not a condensation point of M, and hence we may choose an open set $U \ni y$ such that U is countable. Then $U \cap F_1$ is countable and thus $h^{-1}(U) \cap F \ni x$ is open and countable, contradicting $x \in F$. □

Exercise 1.1.53. ◇ *Let (E, ρ) be a compact metric space, $M = F \cup H$ be as in Proposition 1.1.52, and $g : M \to M$ be a continuous map. Suppose there exists a positive integer L such that, for all $x \in M$, $g^{-1}(x)$ has no more than L elements. Show $g(F) \subset F$.*

The next theorem is a special case of the Baire Category Theorem, namely, when the space involved is a compact metric space; see [55, Theorem D.37] and [170, Problem 24B-4, Section 25].

Theorem 1.1.54. *Let E be a compact metric space. Then E cannot be written as the union of a sequence of nowhere dense sets. Moreover, if $\{A_n\}$ is a sequence of nowhere dense subsets of E, then $E \setminus (\cup_n A_n)$ is dense in E.*

1.2 The Middle Third Cantor Set

Cantor sets arise often in dynamics. As a first example of a Cantor set, we construct the *Middle Third Cantor set.* Some notation: For each $m \in N$ we let $\{0,1\}^m$ denote the 2^m finite strings or words of length m on the alphabet $\{0,1\}$. Similarly, $\{0,1\}^{\mathbb{N}}$ denotes collection of all one-sided infinite strings of 0's and 1's. For $\gamma = \langle \gamma_1, \gamma_2, \gamma_3, \dots \rangle \in \{0,1\}^{\mathbb{N}}$, let $\gamma|n = \langle \gamma_1, \dots, \gamma_n \rangle$.

Middle Third Cantor Set: We recursively define sets K_γ for each $\gamma \in \cup_{m \geq 1} \{0,1\}^m$. Set

$$K_0 = [0, \tfrac{1}{3}] \quad \text{and} \quad K_1 = [\tfrac{2}{3}, 1].$$

If $m \geq 1$ and $K_\gamma = [a, b]$ for $\gamma \in \{0,1\}^m$ are given, set

$$K_{\gamma 0} = [a, a + \frac{1}{3^{m+1}}] \quad \text{and} \quad K_{\gamma 1} = [b - \frac{1}{3^{m+1}}, b].$$

Lastly set

$$K = \cap_{m \geq 1} \ \cup_{\gamma \in \{0,1\}^m} K_\gamma.$$

The Middle Third Cantor set is K. See Figure 1.1.

Exercise 1.2.1. *Prove each of the following.*

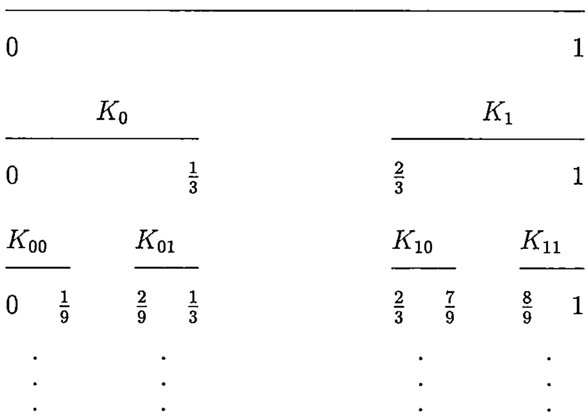

Figure 1.1: Construction of the Middle Third Cantor set

1. *For* $\gamma \in \{0,1\}^m$, *we have* $|K_\gamma| = \frac{1}{3^m}$ *and hence*

$$\sum_{\gamma \in \{0,1\}^m} |K_\gamma| = (\frac{2}{3})^m.$$

2. $\cup_{\gamma \in \{0,1\}^{m+1}} K_\gamma \subset \cup_{\gamma \in \{0,1\}^m} K_\gamma$.

3. $K \neq \emptyset$.

4. *K is closed.*

5. *K contains no nondegenerate open intervals; hence it follows that K is nowhere dense.*

6. *K is perfect.*

7. *K is compact.*

8. *K is totally disconnected.*

9. *The Lebesgue measure of K (see Chapter 2 for a definition of Lebesgue measure) is zero.*

Exercise 1.2.2. *Each $x \in [0,1]$ has an expansion, $\langle x_1, x_2, x_3, x_4, \ldots \rangle$, in ternary form (i.e., $x_i \in \{0,1,2\}$ for all i) obtained by writing $x = \sum_{i \geq 1} \frac{x_i}{3^i}$. These expressions are unique, except that numbers (other than 1) expressible in a ternary expansion ending in a sequence of 2's can be reexpressed in an*