Chapter 1

Operators and Hardy spaces

In this chapter we begin by reviewing the main definitions and theorems from the basic theory of linear operators that will be needed. This material is very standard and likely to have been met in any basic course on functional analysis, and so we give just the essentials of the subject, without proofs.

We then move on to a discussion of the Hardy spaces, which are Banach spaces of functions that can be defined either in the unit disc $\mathbb{D}$ or the right half-plane $\mathbb{C}_+$ and extended, respectively, to the unit circle $\mathbb{T}$ or the imaginary axis $i\mathbb{R}$. Here we give a fairly elementary account of those parts of the subject that are most useful in applications, including the concepts of inner and outer functions. There are many more detailed treatments available, and we refer the interested reader to the notes at the end of the chapter.

1.1 Banach spaces and bounded operators

We shall work with normed spaces, which can be real or complex, and write $\mathbb{K}$ for the field of scalars ($\mathbb{R}$ or $\mathbb{C}$) when it is not important which we take. A complete normed space is called a *Banach space*. An inner product on a vector space induces a norm by means of the formula $\|x\| = \langle x, x\rangle^{1/2}$, and a complete inner-product space is called a *Hilbert space*.

A *linear operator* T from a normed space $\mathcal{X}$ to a normed space $\mathcal{Y}$ is just a linear mapping, that is, it satisfies

$$T(a_1x_1 + a_2x_2) = a_1Tx_1 + a_2Tx_2 \qquad \text{for all } x_1, x_2 \in \mathcal{X} \quad \text{and } a_1, a_2 \in \mathbb{K}.$$

The operator T is said to be *bounded*, if there is a constant $K > 0$ such that $\|Tx\| \le K\|x\|$ for all vectors x in $\mathcal{X}$. The least such K that holds for all x is the *norm* of T, written

$$\|T\| = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\|,$$

where the last equality holds because $\|Tx\|/\|x\| = \|T(x/\|x\|)\|$ and $x/\|x\|$ is a vector of norm 1. The boundedness condition implies that $\|Tx - Ty\| \le K\|x-y\|$ for all $x, y \in \mathcal{X}$, and hence T is continuous. Conversely, continuous operators are always bounded (see the exercises).

The bounded operators from $\mathcal{X}$ to $\mathcal{Y}$ form a normed space, with the norm defined as above, which we shall denote by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$. If $\mathcal{Y}$ is a Banach space, then so is $\mathcal{L}(\mathcal{X}, \mathcal{Y})$. Two special cases are of interest:

1. We denote $\mathcal{L}(\mathcal{X}, \mathcal{X})$ by $\mathcal{L}(\mathcal{X})$. Apart from being a normed space, this is also an algebra, since we can define the product ST of two operators S and T by $(ST)(x) = S(Tx)$ for $x \in \mathcal{X}$; then $\|ST\| \le \|S\|\,\|T\|$.

2. The space $\mathcal{L}(\mathcal{X}, \mathbb{K})$ is the *dual space* of $\mathcal{X}$, denoted by $\mathcal{X}^*$. Its elements are the *linear functionals* on $\mathcal{X}$.

In particular, the dual space of a Hilbert space $\mathcal{H}$ can be identified with the space itself, because every linear functional $f : \mathcal{H} \to \mathbb{K}$ is given by the formula $f(x) = \langle x, y\rangle$ for some unique $y \in \mathcal{H}$. Moreover, $\|f\| = \|y\|$. This is the *Riesz–Fréchet theorem.*

Now let $\mathcal{X}$ be a complex Banach space. For $T \in \mathcal{L}(\mathcal{X})$, the *spectrum* of T is the set

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}.$$

It is known that $\sigma(T)$ is a non-empty compact subset of $\mathbb{C}$ and that, letting $r(T)$ denote the *spectral radius*, $\sup\{|\lambda| : \lambda \in \sigma(T)\}$, we have

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n} = \inf\{\|T^n\|^{1/n} : n \ge 1\}.$$

In particular, $r(T) \le \|T\|$. We also have $\sigma(T) \supseteq \sigma_p(T)$, where $\sigma_p(T)$ denotes the point spectrum of T, the set of eigenvalues of T. If $\mathcal{X}$ is finite-dimensional, these two sets coincide and are finite and non-empty; however, in the infinite-dimensional case, they can be very different (see the exercises).

The *resolvent set*, $\rho(T)$, is the complement of $\sigma(T)$ in $\mathbb{C}$, that is, the set of points $\lambda \in \mathbb{C}$ for which $(T - \lambda I)^{-1}$ exists. We also refer to $(T - \lambda I)^{-1}$ as the *resolvent* of T, which can be regarded as an operator-valued function $R : \rho(T) \to \mathcal{L}(\mathcal{X})$, with $R(\lambda) = (T - \lambda I)^{-1}$.

For a bounded operator $T : \mathcal{H} \to \mathcal{K}$ between Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, the *adjoint* $T^* : \mathcal{K} \to \mathcal{H}$ is defined by the equation

$$\langle Th, k\rangle = \langle h, T^*k\rangle \qquad \text{for all } h \in \mathcal{H} \text{ and } k \in \mathcal{K}. \tag{1.1}$$

The following properties are well known and not difficult to prove. They hold for all T, T_1 and T_2 in $\mathcal{L}(\mathcal{H}, \mathcal{K})$ and a_1, a_2 in $\mathbb{C}$:

- $(a_1T_1 + a_2T_2)^* = \overline{a_1}T_1^* + \overline{a_2}T_2^*$;
- $(T^*)^* = T$;
- $\|T^*\| = \|T\|$;
- $(T_1T_2)^* = T_2^*T_1^*$.

Now three special classes of operator are of interest to us:

1. The operator T is *Hermitian*, or *self-adjoint*, if $T = T^*$.
2. T is *unitary*, if $T^* = T^{-1}$, that is, $TT^* = T^*T = I$.
3. T is *normal*, if $TT^* = T^*T$. Clearly both Hermitian and unitary operators are normal.

If T is Hermitian, then $\sigma(T) \subset \mathbb{R}$; whereas, if T is unitary, then $\sigma(T) \subseteq \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Suppose that $\mathcal{K}$ is a closed subspace of a Hilbert space $\mathcal{H}$. Then $\mathcal{H}$ has an orthogonal decomposition $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^\perp$, where $\mathcal{K}^\perp$, the *orthogonal complement* of $\mathcal{K}$, is the closed subspace $\mathcal{K}^\perp = \{x \in \mathcal{H} : \langle x, k\rangle = 0 \text{ for all } k \in \mathcal{K}\}$. Thus every vector $y \in \mathcal{H}$ decomposes uniquely as $y = k + k'$, where $k \in \mathcal{K}$ and $k' \in \mathcal{K}^\perp$, and one has $\|y\|^2 = \|k\|^2 + \|k'\|^2$.

If we now define $P_\mathcal{K} : \mathcal{H} \to \mathcal{H}$ by $P(k + k') = k$, then $P_\mathcal{K}$ is a linear operator on $\mathcal{H}$, the *orthogonal projection* onto K, and it satisfies $P_\mathcal{K} = P_\mathcal{K}^*$, $P_\mathcal{K} = P_\mathcal{K}^2$ and $\|P_\mathcal{K}\| = 1$ (unless $\mathcal{K} = \{0\}$, when of course $P_\mathcal{K}$ is the zero operator). Moreover, $P_\mathcal{K} + P_{\mathcal{K}^\perp} = I$, the identity operator.

An operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ between Banach spaces is said to be *compact* if T maps bounded subsets of $\mathcal{X}$ into relatively compact subsets of $\mathcal{Y}$ (that is, sets with compact closure). In particular, finite-rank operators are compact. Equivalently, T is compact if, whenever (x_n) is a bounded sequence in $\mathcal{X}$, the sequence (Tx_n) has a convergent subsequence in $\mathcal{Y}$.

The spectrum of a compact operator is particularly simple. It consists of a finite or countably infinite number of points; and if there are infinitely many, they form a sequence tending to zero. All non-zero points of the spectrum are eigenvalues, and the eigenspaces are finite-dimensional.

Suppose now that $T \in \mathcal{L}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. If T is both compact and normal, then T can be decomposed in terms of its non-zero eigenvalues (λ_k)

and eigenvectors (e_k), which can be taken to be an orthonormal sequence as follows:

$$Tx = \sum_k \lambda_k \langle x, e_k \rangle e_k, \qquad (x \in \mathcal{H}).$$

The eigenvalues tend to zero if there are infinitely many of them. Indeed, all operators of this form are both compact and normal.

A general compact operator $T \in \mathcal{L}(\mathcal{H})$ has the following *singular value decomposition*;

$$Tx = \sum_k \sigma_k \langle x, e_k \rangle f_k, \qquad (x \in \mathcal{H}), \tag{1.2}$$

where now (e_k) and (f_k) are two orthonormal sequences, possibly finite, the *Schmidt pairs* of T, and the constants σ_k are positive real numbers, the *singular values* of T, and form a decreasing sequence: if there are infinitely many, they tend to zero. Indeed, (σ_k^2) are the eigenvalues of the compact operator T^*T. In particular, every compact operator on a Hilbert space is the norm limit of a sequence of finite-rank operators – just truncate the sum in (1.2).

1.2 Hardy spaces on the disc and half-plane

We begin by defining the Hardy spaces as Banach spaces of analytic functions on the disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and then see them in another light as spaces of functions defined on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, which we equip with normalized Lebesgue measure.

Definition 1.2.1 *For $1 \le p < \infty$, the Hardy space H^p is defined as the space of all analytic functions f in the disc $\mathbb{D}$ for which the norm*

$$\|f\|_p = \sup_{r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\omega})|^p \, d\omega \right)^{1/p}$$

is finite. The space H^∞ consists of all bounded analytic functions f in the disc with norm

$$\|f\|_\infty = \sup_{|z|<1} |f(z)|.$$

It is not hard to see that for $p \le q$ we have $H^p \supseteq H^q$; thus $H^\infty \subseteq H^2 \subseteq H^1$.

The following result holds, although we shall not require its full power and will prove slightly simpler results for $p = 2$ and $p = \infty$.

Theorem 1.2.2 *For functions f in H^p with $1 \le p \le \infty$, the radial limit*

$$\tilde{f}(e^{i\omega}) = \lim_{r \to 1} f(re^{i\omega})$$

exists almost everywhere in t, and indeed $\tilde{f} \in L^p(\mathbb{T})$, with $\|f\|_{H^p} = \|\tilde{f}\|_{L^p}$.

We normally identify f with $\tilde{f}$ and can thus regard H^p as a closed subspace of $L^p(\mathbb{T})$, and hence a Banach space.

It is also possible to start by defining H^p directly as the subspace of those $L^p(\mathbb{T})$ functions for which the negative Fourier coefficients vanish, that is,

$$\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{f}(e^{i\omega})e^{-in\omega}\, d\omega = 0 \qquad (n < 0).$$

Then a function $\tilde{f}$ with $\tilde{f}(e^{i\omega}) \sim \sum_{n=0}^{\infty} \widehat{f}(n)e^{in\omega}$ can be naturally identified with the power series $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n)z^n$, defining an analytic function f in $\mathbb{D}$.

Proof of part of Theorem 1.2.2: We begin with $p = 2$. For a function $f : z \mapsto \sum_{n=0}^{\infty} a_n z^n$, it is easily verified that we have

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\omega})|^2\, d\omega = \sum_{n=0}^{\infty} r^{2n}|a_n|^2,$$

and thus $f \in H^2$ if and only if $\|f\|_2^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty$. It is also clear that the functions $f_r \in L^2(\mathbb{T})$ defined by $f_r(e^{i\omega}) = f(re^{i\omega})$ converge in the L^2 norm as $r \to 1$ to the function $\tilde{f}$ with Fourier coefficients $(a_n)_{n\geq 0}$, and hence a subsequence converges pointwise almost everywhere. Conversely, any function $\tilde{f} \in L^2(\mathbb{T})$ whose Fourier coefficients of negative index all vanish corresponds in an obvious way to a function f in H^2.

For the case $p = \infty$, we note that $H^\infty \subset H^2$, and thus a function $f \in H^\infty$ also corresponds to a boundary function $\tilde{f} \in L^2(\mathbb{T})$. Because a subsequence of (f_r) tends pointwise almost everywhere to $\tilde{f}$, we may conclude that $\|\tilde{f}\|_\infty \leq \|f\|_\infty$. However, one can obtain the extension from $\tilde{f}$ to f by integrating with the *Poisson kernel* K_r, namely,

$$f(re^{it}) = \frac{1}{2\pi}\int_0^{2\pi} K_r(t-\omega)\tilde{f}(e^{i\omega})\, d\omega,$$

where

$$K_r(t-\omega) = \frac{1-r^2}{1+r^2-2r\cos(t-\omega)} = \mathrm{Re}\left(\frac{e^{i\omega}+re^{it}}{e^{i\omega}-re^{it}}\right). \tag{1.3}$$

This implies that

$$\|f\|_\infty \leq \sup_{0\leq r<1} \|\tilde{f}\|_\infty \|K_r\|_1 = \|\tilde{f}\|_\infty,$$

since the Poisson kernel is a positive function and

$$\|K_r\|_1 = \frac{1}{2\pi}\int_0^{2\pi} K_r(t)\, dt = 1 \qquad (0 \leq r < 1).$$

We thus have $\|\tilde{f}\|_\infty = \|f\|_\infty$. □

The Poisson kernel above provides a harmonic extension to the disc for any function in $L^1(\mathbb{T})$: the function is in the Hardy class if and only if this extension is actually an analytic function in $\mathbb{D}$.

We see that H^2 is a Hilbert space, being a closed subspace of the Hilbert space $L^2(\mathbb{T})$, and we shall use P_{H^2} to denote the orthogonal projection from $L^2(\mathbb{T})$ onto H^2, so that

$$P_{H^2} : \sum_{n=-\infty}^{\infty} a_n e^{in\omega} \mapsto \sum_{n=0}^{\infty} a_n e^{in\omega}.$$

There is a natural isometric isomorphism between the sequence space $\ell^2(\mathbb{Z})$ and the function space $L^2(\mathbb{T})$, given by associating the sequence $(a_n)_{n=-\infty}^{\infty}$ with the function whose Fourier series is $\sum_{n=-\infty}^{\infty} a_n e^{in\omega}$. Under this correspondence, the sequence space $\ell^2(\mathbb{Z}_+)$ maps to the Hardy space H^2, for we may regard $\ell^2(\mathbb{Z}_+)$ as embedding into $\ell^2(\mathbb{Z})$ as the subspace of all $(a_n)_{n=-\infty}^{\infty}$ with $a_n = 0$ for $n < 0$.

The space H^2 is also a *reproducing kernel Hilbert space* on the disc $\mathbb{D}$. What this means is that the evaluation functional $f \mapsto f(a)$ is bounded for each $a \in \mathbb{D}$. By the Riesz–Fréchet theorem given in Section 1.1, we can find a function $k_a \in H^2$, the *reproducing kernel*, such that

$$f(a) = \langle f, k_a \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\omega}) \overline{k_a(e^{i\omega})}\, d\omega,$$

and in fact in this case $k_a(z) = 1/(1 - \overline{a}z)$.

Another useful result is the following, which describes the boundary behaviour of an H^p function. We shall omit the proof, although in Chapter 3 we shall see that the final statement can be proved using the theory of shift-invariant subspaces (see Exercise 6 of Chapter 3).

Theorem 1.2.3 *Suppose that $f \in H^p$ for some $p \geq 1$, and that f is not identically zero. Then*

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\omega})|\, d\omega > -\infty,$$

and hence $f(e^{i\omega}) \neq 0$ almost everywhere.

We now present the analogous results for Hardy spaces defined on the right half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

Definition 1.2.4 *For $1 \leq p < \infty$, the* Hardy space $H^p(\mathbb{C}_+)$ *of the right half-plane $\mathbb{C}_+$ may be defined as the set of all analytic functions $f : \mathbb{C}_+ \to \mathbb{C}$ such* *that*

$$\|f\|_p = \left(\sup_{x>0} \int_{-\infty}^{\infty} |f(x+iy)|^p \, dy \right)^{1/p} < \infty.$$

Likewise, the space $H^\infty(\mathbb{C}_+)$ *consists of all analytic and bounded functions in* $\mathbb{C}_+$*, and the norm is given by*

$$\|f\|_\infty = \sup_{z \in \mathbb{C}_+} |f(z)|.$$

Again these functions have boundary values $\tilde{f}(iy) = \lim_{x \to 0+} f(x+iy)$ almost everywhere, and the boundary function $\tilde{f}$ lies in $L^p(i\mathbb{R})$ and satisfies $\|\tilde{f}\|_{L^p} = \|f\|_{H^p}$. We may identify f and $\tilde{f}$, and thus $H^p(\mathbb{C}_+)$ can naturally be regarded as a closed subspace of $L^p(i\mathbb{R})$ and hence a Banach space.

As in the case of the disc, a Poisson kernel formula can be used to provide harmonic extensions to the right half-plane of functions lying in $L^p(i\mathbb{R})$ for some $1 \le p \le \infty$, and the H^p functions are those whose harmonic extensions are analytic. The formula in this case is

$$f(x+iy) = \frac{1}{\pi} \int_{-\infty}^{\infty} \hat{K}_x(y-t) \tilde{f}(it) \, dt, \qquad \text{for } x > 0,$$

where $\hat{K}_x$ is the *Poisson kernel for* $\mathbb{C}_+$ and is given by

$$\hat{K}_x(y-t) = \frac{x}{x^2 + (y-t)^2}.$$

The *Laplace transform* $L : L^2(0,\infty) \to H^2(\mathbb{C}_+)$ plays an important role here. It is given by

$$(Lg)(s) = \int_0^\infty e^{-st} g(t) \, dt \qquad (s \in \mathbb{C}_+)$$

and up to a constant factor gives an isometric isomorphism between the two spaces, since it is bijective and satisfies $\|Lg\|_{H^2} = \sqrt{2\pi}\|g\|_{L^2}$. What is even more remarkable is the content of the Paley–Wiener theorem, namely, that up to a change of variable we may use the bilateral Laplace transform (which is the Fourier transform with a change of variable) to decompose $L^2(i\mathbb{R})$ into the orthogonal sum of Hardy spaces on the left and right half-planes. Explicitly, if we now write

$$(Lg)(s) = \int_{-\infty}^{\infty} e^{-st} g(t) \, dt$$

for $s \in i\mathbb{R}$ and $g \in L^1(\mathbb{R})$, this extends by continuity to a linear isomorphism $L : L^2(\mathbb{R}) \to L^2(i\mathbb{R})$ with $\|Lg\|_{L^2} = \sqrt{2\pi}\|g\|_{L^2}$, and applying L to the decomposition

$$L^2(\mathbb{R}) \;=\; L^2(-\infty, 0) \oplus L^2(0,\infty)$$

gives

$$L^2(i\mathbb{R}) \quad = \quad H^2(\mathbb{C}_-) \oplus H^2(\mathbb{C}_+),$$

where $\mathbb{C}_-$ and $\mathbb{C}_+$ are, respectively, the left and right half-planes. We identify functions with their harmonic extensions, as before.

We note that $H^2(\mathbb{C}_+)$ is again a reproducing kernel Hilbert space, in that, for $s \in \mathbb{C}_+$ and $f \in H^2(\mathbb{C}_+)$, we have

$$f(s) = \langle f, k_s \rangle = \int_{-\infty}^{\infty} f(iy)\overline{k_s(iy)}dy,$$

where $k_s(z) = \dfrac{1}{2\pi(z+\bar{s})}$ is the reproducing kernel for $H^2(\mathbb{C}_+)$.

There is a natural isometric isomorphism between Hardy spaces on the disc and half-plane, which is induced by the self-inverse bijection $M : \mathbb{D} \to \mathbb{C}_+$, given by $M(z) = (1-z)/(1+z)$. The following relation is given in [57, 97], for example.

Theorem 1.2.5 *The mapping $V : H^2(\mathbb{D}) \to H^2(\mathbb{C}_+)$, defined by*

$$(Vf)(s) = \frac{1}{\sqrt{\pi}(1+s)} f(M(s)),$$

is an isometric isomorphism.

1.3 Inner and outer functions

In this section we are concerned with the multiplicative structure of the Hardy spaces, in that we want to factorize a general Hardy class function as the product of two somewhat simpler functions, an inner factor and an outer factor. Here are their definitions (a simpler characterization of outer functions appears as a corollary of Beurling's theorem, in Corollary 3.1.4).

Definition 1.3.1 *An* inner function *is an H^∞ function that has unit modulus almost everywhere on $\mathbb{T}$. An* outer function *is a function $f \in H^1$ that can be written in the form*

$$f(z) = \alpha \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\omega}+z}{e^{i\omega}-z}\, k(e^{i\omega})\, d\omega\right) \qquad (z \in \mathbb{D}), \tag{1.4}$$

where k is a real-valued integrable function and $|\alpha| = 1$.

If f is an outer function satisfying (1.4), then $\log|f(e^{it})| = k(e^{it})$ almost everywhere, since $\log|f(re^{it})| = \operatorname{Re}\log f(re^{it})$ gives the Poisson extension of k to the disc by virtue of (1.3). Clearly, an outer function can have no zeroes in the disc, since it is the exponential of something.

Example 1.3.2 *Examples of outer functions include all invertible functions in H^∞ (for example, polynomials whose zeroes all lie outside $\overline{\mathbb{D}}$). In fact it can be shown that all polynomials whose zeroes lie in $\mathbb{C}\setminus\mathbb{D}$ are outer functions, although they are not invertible in H^∞ if they have zeroes on the unit circle.*

Examples of inner functions include functions such as $z \mapsto \dfrac{z-a}{1-\overline{a}z}$, where $a \in \mathbb{D}$, and, more generally, Blaschke products (see Definition 1.3.4 below); these have zeroes in the disc, but there are also inner functions without zeroes, such as $\exp((z-1)/(z+1))$. This last function is just e^{-s}, where $s = (1-z)/(1+z)$; the mapping from z to s takes $\mathbb{D}$ to the right half-plane $\mathbb{C}_+$ and $\mathbb{T}\setminus\{-1\}$ to the imaginary axis $i\mathbb{R}$.

Once we have a complete characterization of inner functions and outer functions, we have a full description of all Hardy class functions, because of the following factorization theorem, which decomposes an arbitrary function into the product of inner and outer factors.

Theorem 1.3.3 (Inner–outer factorization) *Let f be a nonzero function in H^1. Then f has a factorization $f = \theta \cdot u$, where θ is inner and u is outer. This factorization is unique up to a constant of modulus 1.*

Sketch proof: Since $f \in H^1$, the function $\log|f|$ lies in $L^1(\mathbb{T})$, by virtue of Theorem 1.2.3. We can thus define the outer factor u corresponding to f by the formula

$$u(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\omega}+z}{e^{i\omega}-z}\log|f(e^{i\omega})|\,d\omega\right), \qquad (z \in \mathbb{D}),$$

after which $\theta = f/u$ is analytic in $\mathbb{D}$ with boundary values of modulus 1 almost everywhere, and thus θ is inner. The uniqueness of the decomposition follows on observing that a unimodular outer function is necessarily constant. □

We are going to see a complete description of the class of inner functions, and we begin with those that have zeroes in $\mathbb{D}$.

Definition 1.3.4 *A* finite Blaschke product *is a function of the form*

$$B(z) = \alpha\prod_{j=1}^{n}\frac{z-z_j}{1-\overline{z}_j z},$$

where $|\alpha| = 1$ and $|z_j| < 1$ for $j = 1, \ldots, n$.

It is easy to verify that B is analytic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$, that B is inner, and that B has zeroes at $z_1, \ldots, z_n$ only and poles at $1/\overline{z}_1, \ldots, 1/\overline{z}_n$ only.

Next we want to break the inner part into two factors, an inner function with zeroes (which will be an infinite Blaschke product) and an inner function without zeroes (a so-called *singular inner function*). To do this we need to understand the properties of the zero set of a function in H^p.

Theorem 1.3.5 (G. Szegő) *Let $f \in H^1$ be such that f is not identically zero. Then the zeroes (z_n) of f are countable in number and satisfy the Blaschke condition*

$$\sum_1^\infty (1 - |z_n|) < \infty. \tag{1.5}$$

Proof: By considering $z \mapsto f(z)/z^j$, if necessary, we may suppose without loss of generality that $f(0) \neq 0$. Now take $0 < r < 1$ and let $z_1, \ldots, z_m$ be the zeroes of f in $\{z \in \mathbb{C} : |z| < r\}$, choosing r so that there are none on $\{|z| = r\}$. Write $g(z) = f(rz)/B(z)$, where B is a Blaschke product with zeroes $z_1/r, \ldots, z_m/r$. Since $\log g$ is harmonic, we have the identity

$$\log g(0) = \frac{1}{2\pi} \int_0^{2\pi} \log g(e^{i\omega})\, d\omega,$$

which, on taking real parts, reduces to

$$\log |f(0)| + \sum_{|z_n|<r} \log(r/|z_n|) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\omega})|\, d\omega \le \log \|f\|_1,$$

by Jensen's inequality $\int \phi(f(x))\, dx \le \phi \int f(x)\, dx$ holding for concave functions ϕ – in this case $\phi(y) = \log(y)$.

Letting $r \to 1$, we see that $\sum_n \log(1/|z_n|) < \infty$, which, by the comparison test, is easily seen to be equivalent to $\sum (1 - |z_n|) < \infty$. □

For a Hardy class function f we can construct a Blaschke product B whose zeroes are precisely the zeroes of f; then f/B has no zeroes at all and can be analysed further.

Theorem 1.3.6 *Let $f \in H^1$. Then the infinite Blaschke product*

$$B(z) = z^m \prod_{|z_n| \neq 0} \frac{\overline{z}_n}{|z_n|} \frac{z_n - z}{1 - \overline{z}_n z},$$

where (z_n) are the zeroes of f, of which m are at 0, converges uniformly on compact sets to an H^∞ function, the only zeroes of which are the (z_n), with the correct multiplicities. Moreover, $|B(z)| \le 1$ and $|B(e^{i\omega})| = 1$ almost everywhere.