Cambridge University Press
978-0-521-54547-1 — Transcendental Aspects of Algebraic Cycles
Edited by S. Müller-Stach , C. Peters
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Part I

Introductory material

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Chow varieties, the Euler–Chow series and the total coordinate ring

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Introduction

Chow varieties play an important role in the geometry and topology of algebraic varieties. However, their geometry and topology are not well understood. It is important to mention the work of Blaine Lawson, Eric Friedlander, Paulo Lima-Filho and others on the homotopy and topology of the space of cycles with fixed dimension. Chapter 3 by Lima-Filho is relevant in this aspect.

In this article we would like to present other aspects of the geometry and topology of Chow varieties.

In Section 1.1 we introduce Chow varieties and give some important examples. In the last part of this section we mention the case of zero cycles, and we state a theorem where it is shown that they are isomorphic to a certain symmetric product.

In Section 1.2 we study the Euler–Chow series. These are a class of invariants for projective varieties arising from the Euler characteristic of their Chow varieties. It is a series that in a way generalizes the Hilbert series and also appears in many different problems in algebraic geometry. It is also worth mentioning that it belongs, for the correct dimension, as an element in the quantum cohomology of the variety. We do not know what role it plays here, but it shows that a lot more has to be understood before we can get a clear picture of the role of this series in geometry.

In this section we start with simple and interesting examples, then we go through some formalism in order to understand the product of two Euler–Chow series of different varieties. Then some properties of the series are shown and the series is computed for some important examples, like toric varieties, abelian varieties, projective closure of line bundles, and other cases. In Section 1.2.4

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we relate the Euler–Chow series for the Grassmannian varieties with the Chow quotients, another interesting fact that perhaps shows how the series is full of information about the geometry of the variety.

In Section 1.3 we state some open problems and introduce the total coordinate ring of a variety, stating some theorems. This ring is associated to the Euler–Chow series of a projective variety and only for divisors. The ring itself is very interesting; it is related to different classical problems in algebraic geometry, canonical rings and Mori theory, an old classical problem by Zariski, and some others.

1.1 Chow varieties

In this section we will sketch the construction of Chow varieties and give some examples. There are two main references, the first of which is the book of Shafarevich [Sha77]. It is important to note that the last edition which consists of two volumes does not have the construction of the Chow varieties. The second main reference where most of the examples and material can be found is the book of Gelfand, Kapranov and Zelevinsky [GKZ94]. The reader is encouraged to consult the latter for details and proofs of some of the theorems.

We should also mention the book of Kollár [Kol96]; it has an excellent exposition of Hilbert schemes and Chow varieties in characteristic p. Although this is very important, it is not possible to cover it in these notes. Throughout this section we work over an algebraic closed field.

1.1.1 Chow forms of irreducible varieties

Let $X \subset \mathbb{P}^{n-1}$ be an irreducible subvariety of dimension $k-1$ and degree d. Let $\mathcal{Z}(X)$ be the set of all $(n-k-1)$-dimensional projective subspaces $L \in \mathbb{P}^{n-1}$ that intersect X. This is a subvariety in the Grassmannian $G(n-k, n)$ parametrizing all the $(n-k-1)$-dimensional projective subspaces in $\mathbb{P}^{n-1}$. Then we have the following theorem.

Theorem 1.1.1.1 *The subvariety $\mathcal{Z}(X)$ is an irreducible hypersurface of degree d in $G(n-k, n)$.*

We shall call $\mathcal{Z}(X)$ the *associated hypersurface* of X. Let $\mathfrak{B} = \oplus \mathfrak{B}_m$ be the coordinate ring of the Grassmannian $G(n-k, n)$. It can be proven that $\mathcal{Z}(X)$ is defined by the vanishing of some element $R_X \in \mathfrak{B}_d$. This element is called the *Chow form* of X. The coefficients of this form are the *Chow coordinates* of X. It is important to notice that X can be recovered from its Chow coordinates. Consider these examples.

Example 1.1.1.2

1. Let X be a curve in $\mathbb{P}^3$. Its associate hypersurface is the variety of all lines which intersect X.
2. Let X be a hypersurface in $\mathbb{P}^{n-1}$. The Grassmannian $G(n-k, n)$ coincides with $\mathbb{P}^{n-1}$ and the associated hypersurface $\mathcal{Z}(X)$ coincides with X.
3. Let X be a point p, then $G(n-k, n)$ is the dual projective space $(\mathbb{P}^{n-1})^*$ and $\mathcal{Z}(X)$ is the hyperplane dual to p.
4. Let X be a linear projective subspace. The variety $\mathcal{Z}(X)$ is known as the *Schubert divisor* in $G(n-k, n)$. We can assume that the linear equations of X are given by $x_1 = 0, \ldots, x_{n-k} = 0$ where $x_1, \ldots, x_n$ are coordinate functions. Then the associate variety is given by the vanishing of the Plücker coordinates $p_1, \ldots, p_{n-k}$. For explicit formulas of the Plücker coordinates, see [Ful98].

Now, we have the following theorem that tells us that we can recover X from $\mathcal{Z}(X)$.

Theorem 1.1.1.3 *A $(k-1)$-dimensional irreducible subvariety $X \subset \mathbb{P}^{n-1}$ is determined uniquely by its associated hypersurface $\mathcal{Z}(X)$. More precisely, a point $p \in \mathbb{P}^{n-1}$ lies in X if and only if any $(n-k-1)$-dimensional plane containing p belongs to $\mathcal{Z}(X)$.*

1.1.2 Definition of Chow variety

In this section we construct for any irreducible $(k-1)$-dimensional subvariety $X \subset \mathbb{P}^{n-1}$, its Chow form R_X. This is a polynomial $R_X(f_1, \ldots, f_k)$ in coefficients of k indeterminate linear forms on $\mathbb{C}^n$ which vanishes whenever the projective subspace $\{f_1 = \cdots = f_k = 0\}$ of $\mathbb{P}^{n-1}$ intersects X. It also satisfies the following homogeneity property, if $g = (g_{ij})$ is a matrix in $GL(k)$, we have that

$$R_X(g_{11} f_1 + \cdots + g_{1k} f_k, \ldots, g_{k1} f_1 + \cdots + g_{kk} f_k) \\ = \det(g)^d \, R_X(f_1, \ldots, f_k)$$

where $d = \deg X$. The space of polynomials with this property is denoted by $\mathfrak{F}_d$.

Let $X = \sum m_i X_i$ be a $(k-1)$-dimensional effective algebraic cycle in $\mathbb{P}^{n-1}$ of degree d. We define the Chow form of X as

$$R_X = \prod R_{X_i}^{m_i} \in \mathfrak{F}_d.$$

The coordinates of the vector R_X are called *Chow coordinates* of X. Let us denote by $\mathcal{C}_{k-1,d}\left(\mathbb{P}^{n-1}\right)$ the space of all the effective $(k-1)$-cycles in $\mathbb{P}^{n-1}$ of degree d. The main result is a theorem due to Chow and van der Waerden.

Theorem 1.1.2.1 *The map $X \longmapsto R_X$ defines an embedding of $\mathcal{C}_{k-1,d}\left(\mathbb{P}^{n-1}\right)$ into the projective space $\mathbb{P}^{\mathfrak{F}_d}$ as a closed algebraic variety.*

The variety $\mathcal{C}_{k-1,d}\left(\mathbb{P}^{n-1}\right)$ with the algebraic structure defined by the above embedding is called the *Chow embedding*. For a proof of this theorem see [GKZ94, p. 126].

1.1.3 Examples of Chow varieties

Example 1.1.3.1

1. The Chow variety $\mathcal{C}_{k-1,1}\left(\mathbb{P}^{n-1}\right)$ is the Grassmannian $G(k,n)$ and its Chow embedding coincides with the Plücker embedding.
2. Consider the Chow variety $\mathcal{C}_{n-2,d}\left(\mathbb{P}^{n-1}\right)$, parametrizing cycles of degree d and codimension 1 in $\mathbb{P}^{n-1}$, that is hypersurfaces. We saw in Example 1.1.1.2 that the Chow form of an irreducible hypersurface is just its equation which is an irreducible homogeneous polynomial of degree d in n variables. Algebraic cycles of codimension 1 corresond to all non-zero homogeneous polynomials, irreducible or not, of degree d. Therefore, the Chow variety $\mathcal{C}_{n-2,d}\left(\mathbb{P}^{n-1}\right)$ is the projective space of such polynomials, i.e.

$$\mathcal{C}_{n-2,d}\left(\mathbb{P}^{n-1}\right) = \mathbb{P}^{N-1} \qquad \text{where} \qquad N = \binom{n+d-1}{d}.$$

Example 1.1.3.2 Consider $\mathcal{C}_{1,2}\left(\mathbb{P}^3\right)$. Thus we are considering curves of degree 2 in $\mathbb{P}^3$. There are two cases, either an irreducible curve or two lines. A curve of degree 2 must be a plane quadric, by Bézout. This implies that $\mathcal{C}_{1,2}\left(\mathbb{P}^3\right)$ has two irreducible components C and D corresponding to planar quadrics and pairs of lines. $C \cap D$ consists of pairs of coplanar lines. Now, $\dim D = 8$ since one line in $\mathbb{P}^3$ depends on four parameters. What is interesting, and rare, in Chow varieties is that $\dim C = 8$. This is easy to see since a plane needs three parameters, and a quadric in a plane needs five parameters.

Example 1.1.3.3 This example is $\mathcal{C}_{1,3}\left(\mathbb{P}^3\right)$ parametrizing 1-dimensional cycles in $\mathbb{P}^3$ of degree 3. The possible curves for those cycles are:

1. an irreducible curve of degree 3; here we have two possible choices (see [Har75b, IV.6]), either a twisted curve or a plane cubic;
2. a line and a planar quadric;
3. three lines.

A twisted cubic is a curve which can be modified by a projective transformation of $\mathbb{P}^3$ to the standard Veronese curve given in homogeneous coordinates by

$$\left\{\left(x_0^3 : x_0^2 x_1 : x_0 x_1^2 : x_1^3\right) \,\middle|\, (x_0 : x_1) \in \mathbb{P}^1\right\}. \tag{1}$$

If we denote by C_1, C_2, C_3, C_4 the subvarieties of $\mathcal{C}_{1,3}\left(\mathbb{P}^3\right)$ parametrizing twisted curves, plane cubics, a line and a planar quadric, and three lines, then we have that

$$\mathcal{C}_{1,3}\left(\mathbb{P}^3\right) = C_1 \cup C_2 \cup C_3 \cup C_4.$$

The dimension of C_4 is 12, since one line depends on four parameters. The dimension of C_3 is also 12; check the last example. The dimension of C_1 is also 12. To see this we observe that all twisted cubics are images of one particular twisted cubic, see equation (1), under projective transformations. The stabilizer of the curve in (1) is the group $PGL(2)$ of projective transformations of $\mathbb{P}^1$ embedded into $PGL(4)$ via the map

$$GL(2) = GL(\mathbb{C}^2) \hookrightarrow GL(4) = GL(S^3\mathbb{C}^2)$$

by the correspondence

$$g \longmapsto S^3 g$$

where $S^3\mathbb{C}^2$ is the space of all homogeneous polynomials of degree d in two variables. Hence, $C_1 = PGL(4)/PGL(2)$, and its dimension is equal to $15 - 3 = 12$.

For C_2, the dimension is given by the number of parameters defining a plane, which is three, plus the dimension of the space of cubics in a given plane, which is nine, therefore, $\dim C_2 = 12$.

It is tempting to conjecture that all components of the variety $\mathcal{C}_{1,d}\left(\mathbb{P}^3\right)$ have dimension $4d$. However, there is this example.

Example 1.1.3.4 Consider the Chow variety $\mathcal{C}_{1,4}\left(\mathbb{P}^3\right)$ of 1-dimensional cycles in $\mathbb{P}^3$ of degree 4. This variety has many components corresponding to the various possibilities that occur for a cycle of degree 4:

1. an irreducible curve of degree 4;
2. a cubic and a line;
3. two quadric curves;
4. a quadric curve and two lines;
5. four lines.

From the previous examples it is reasonably clear that all of the components in cases 2–5 above have dimension 16. Thus, we have to concentrate on the first component, let us call it C. This variety also has reducible components, irreducible curves of degree 4 that can be of three different types (see [Har75b, IV.6]), namely:

1. a planar quartic;
2. a rational curve of degree 4;
3. a spatial elliptic curve of degree 4.

The last one is the intersection of two quadric surfaces. Let C_1, C_2, C_3 be the components corresponding to 1, 2, 3. C_2 and C_3 have dimension 16. However, the number of parameters defining a plane is three, plus the dimension of the space of quartics in a given plane is 14. Therefore $\dim C_1 = 17$.

We mention in passing that Eisenbud and Harris have computed the dimension of the Chow variety of curves [EH92]. A student of Harris, Pablo Azcue, computed in his Ph.D. thesis the dimension of Chow varieties in higher dimensions. In both cases there are small numbers of Chow varieties (of low degree) that cannot be considered by their computations.

1.1.4 Zero cycles

A positive 0-cycle of degree d is just an unordered collection $\{x_1, \ldots, x_d\}$ of d points (not necessarily distinct) in $\mathbb{P}^{n-1}$. Thus, as a set $\mathcal{C}_{0,d}(\mathbb{P}^n)$ is identified with $\mathrm{Sym}^d(\mathbb{P}^{n-1})$, the d-fold symmetric product of $\mathbb{P}^{n-1}$. So we start with a comparison of the dimensions of $\mathcal{C}_{0,d}\left(\mathbb{P}^{n-1}\right)$ and $\mathrm{Sym}^d(\mathbb{P}^{n-1})$.

Suppose that our projective space $\mathbb{P}^{n-1}$ is $P(V)$ where V is an n-dimensional vector space. The Chow form of a point $x \in P(V)$ is the linear function l_x on V^* given by the scalar product with x:

$$l_x(\xi) = (x, \xi)\,.$$

If $X = \sum m_i x_i$ is a positive 0-cycle in $P(V)$ then, by our convention, the Chow form R_X is the polynomial $\xi \mapsto \prod l_{x_i}^{m_i}(\xi)$. We arrive at the following.

Proposition 1.1.4.1 *The Chow variety $\mathcal{C}_{0,d}\left(\mathbb{P}^{n-1}\right)$ of positive 0-cycles in $\mathbb{P}^{n-1}$ of degree d is the projectivization of the space of homogeneous polynomials of degree d in n variables which are products of linear forms.*

The set Y of decomposable (into linear factors) polynomials of degree d was already used several times in the course of proving the Chow–van der Waerden theorem. Note that this set has, as its ‘odd’ analogue, the set of polyvectors from

$\bigwedge^d \mathbb{C}^n$ which are decomposable into wedge products of d vectors. The projectivization of the set of decomposable polyvectors is, as we have seen in Section 1.1.3, nothing more than the Grassmannian $G(d, n)$ in its Plücker embedding. So the variety of 0-cycles is the 'even' analogue of the Grassmannian.

Recall now the definitions of symmetric products. Let X be a quasi-projective algebraic variety. The symmetric product $\mathrm{Sym}^d(X)$ is the quotient of the Cartesian product X^d by the action of the symmetric group S_d permuting the factors. A more precise definition is as follows.

Suppose first that X is an affine variety and R is its coordinate ring. So $R^{\otimes d} = R \otimes \cdots \otimes R$ is the coordinate ring of X^d. The coordinate ring of $\mathrm{Sym}^d(X)$ is, by definition, the subring of S_d-invariants in $R^{\otimes d}$. In other words, this is the ring of regular functions $f(\mathbf{x}_1, \ldots, \mathbf{x}_d)$ of d variables $\mathbf{x}_i \in X$ which are symmetric, i.e. invariant under any permutation of the $\mathbf{x}_i$.

If X is an arbitrary, not necessarily affine, quasi-projective variety then the symmetric product $\mathrm{Sym}^d(X)$ is defined by gluing affine varieties $\mathrm{Sym}^d(U)$ for various affine open subsets $U \subset X$.

It follows from these definitions that we have a regular morphism of algebraic varieties

$$\gamma\colon\ \mathrm{Sym}^d(\mathbb{P}^{n-1}) \to \mathcal{C}_{0,d}\left(\mathbb{P}^{n-1}\right) \qquad \{\mathbf{x}_1, \ldots, \mathbf{x}_d\} \mapsto \sum x_i, \tag{2}$$

which is set theoretically a bijection. Note that this does not automatically imply that γ is an isomorphism of algebraic varieties: the morphism from the affine line A^1 to the cubic $y^2 = x^3$, given by $x(t) = t^2$, $y(t) = t^3$, is bijective but not an isomorphism since the cubic is singular at (0, 0). So the following fact requires a proof.

Theorem 1.1.4.2 *The morphism* $\gamma\colon\ \mathrm{Sym}^d(\mathbb{P}^{n-1}) \to \mathcal{C}_{0,d}\left(\mathbb{P}^{n-1}\right)$ *is an isomorphism of algebraic varieties* (*over the field of complex numbers*).

It is important to note that over a field of finite characteristic the statement is no longer true [Nee91].

Finally, we would like to state two more results.

Theorem 1.1.4.3 *The symmetric product* $\mathrm{Sym}^d(\mathbb{P}^1) = \mathcal{C}_{0,d}\left(\mathbb{P}^1\right)$ *is isomorphic to* $\mathbb{P}^d$.

Theorem 1.1.4.4 *For any d and n, the variety* $\mathrm{Sym}^d(\mathbb{P}^{n-1}) = \mathcal{C}_{0,d}\left(\mathbb{P}^{n-1}\right)$ *is rational, i.e. it is birationally isomorphic to the projective space* $\mathbb{P}^{d(n-1)}$.

1.2 The Euler–Chow series of Chow varieties

1.2.1 General definitions

The use of topological invariants on moduli spaces has played a vital role in various branches of mathematics and mathematical physics in the last two decades. A quick sampling under this vast umbrella includes works in gauge theory, the theory of instantons, various moduli spaces of vector bundles, moduli spaces of curves and their compactifications, Chow varieties and Hilbert schemes.

In this section we study a class of invariants for projective varieties arising from the Euler characteristics of their Chow varieties. We will see that quite nice and elegant behaviour which can often be codified in simple generating functions comes from these Euler characteristics.

Basic examples

In this subsection we follow [ELF98]. As for motivation, we start with some particular cases which are well studied in the literature. Let X be a connected projective variety and let $SP(X)$ denote the disjoint union $\coprod_{d\geq 0} SP_d(X)$ of all symmetric products of X, with the disjoint union topology, where $SP_0(X)$ is a single point. One can define a function $E_0(X) : \mathbb{Z}_+ = \pi_0(SP(X)) \to \mathbb{Z}$ which sends d to the Euler characteristic $\chi(SP_d(X))$ of the d-fold symmetric product of X. This is what we call the 0*th Euler–Chow function of* X. The same information can be codified as a formal power series $E_0(X) = \sum_{d\geq 0} \chi(SP_d(X))t^d$, and a result of Macdonald [Mac62] shows that $E_0(X)$ is given by the rational function $E_0(X) = (1/(1-t))^{\chi(X)}$.

Another familiar instance arises in the case of divisors. Given an n-dimensional projective variety X, let $\mathrm{Div}_+(X)$ denote the space of effective divisors on X and assume that $\mathrm{Pic}^0(X) = \{0\}$. Consider the function $E : \mathrm{Pic}(X) \to \mathbb{Z}$ which sends $L \in \mathrm{Pic}(X)$ to $\dim H^0(X, \mathcal{O}(L))$. Observe that:

1. given $L \in \mathrm{Pic}(X)$, then $E(L) \neq 0$ if and only if $L = \mathcal{O}(D)$ for some effective divisor D;
2. under the given hypothesis, algebraic and linear equivalence coincide, and two effective divisors D and D' are algebraically equivalent if and only if they are in the same linear system.

The last observation implies that $\mathrm{Div}_+(X)$ can be written as $\mathrm{Div}_+(X) = \coprod_{\alpha\in\mathcal{A}^{\geq}_{n-1}(X)} \mathrm{Div}_+(X)_\alpha$, where $\mathcal{A}^{\geq}_{n-1}(X)$ is the monoid of algebraic equivalence classes of effective divisors (cf. [Ful98, Section 12]), and $\mathrm{Div}_+(X)_\alpha$ is the linear system associated to $\alpha \in \mathcal{A}^{\geq}_{n-1}(X)$. The first observation shows that the only data relevant to E is given by $\mathcal{A}^{\geq}_{n-1}(X) \subset \mathrm{Pic}(X)$. Therefore, we might as well restrict E and define the $(n-1)$*-st Euler–Chow function* of X as the function $E_{n-1}(X) : \mathcal{A}^{\geq}_{n-1}(X) \to \mathbb{Z}_+$ which sends $\alpha \in \mathcal{A}^{\geq}_{n-1}(X)$ to the Euler characteristic $\chi(\mathrm{Div}_+(X)_\alpha) = \dim H^0(X, \mathcal{O}(L_\alpha))$, where L_α is the line bundle associated to α.

Example 1.2.1.1 An even more restrictive case arises when $\mathrm{Pic}(X) \cong \mathbb{Z}$, and $\mathcal{A}^{\geq}_{n-1}(X) \cong \mathbb{Z}_+$ is generated by the class of a very ample line bundle L. Then the $(n-1)$-st Euler–Chow function $E_{n-1}(X) = \sum_{d\geq 0} \dim H^0(X; \mathcal{O}(L^{\otimes n}))t^n$ is just the *Hilbert function* associated to the projective embedding of X induced by L. This is once again a rational function.

Preliminary definitions

Let us start with an abelian monoid M, whose multiplication we denote by $*_M : M \times M \longrightarrow M$. When no confusion is likely to arise we use an additive notation $+ : M \times M \longrightarrow M$ with no subscripts attached. We say that M has *finite multiplication* if $*_M$ has finite fibers. Typical examples are the freely generated monoids, such as the non-negative integers $\mathbb{Z}_+$ under addition.

Definition 1.2.1.2 Given a monoid with finite multiplication M, and a commutative ring S, denote by S^M the set of all functions from M to S. If f and f' are elements in S^M, let $f + f' \in S^M$ be defined by pointwise addition, i.e. $(f + f')(m) = f(m) + f'(m)$. Define the product $f * f' \in S^M$ as the 'convolution'

$$(f * f')(m) = \sum_{a *_M b = m} f(a) f'(b).$$

It is easy to see that S^M then becomes a commutative ring with unity, under these operations.

Remark 1.2.1.3 The ring S^M can be identified with the completion $S[\![M]\!]$ of the monoid algebra $S[M]$ at its augmentation ideal. Therefore, the elements of S^M can be written as a formal power series $f = \sum_{m\in M} s_m \cdot t^m$, on variables t^m and coefficients in S. In this form the multiplication is given by the relation $t^m t^{m'} = t^{m+m'}$ for elements $m, m' \in M$.

Definition 1.2.1.4 Given a monoid morphism $\Psi : M \longrightarrow N$, $f \in S^M$ and $g \in S^N$, define $\Psi^\sharp g \in S^M$ and $\Psi_\sharp f \in S^N$ by

$$(\Psi^\sharp g)(m) = g(\Psi(m))$$

and

$$(\Psi_\sharp f)(n) = \sum_{m \in \Psi^{-1}(n)} f(m)$$

if Ψ has finite fibers.