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# 1. A Few Noetherian Rings

After a review of the definition and basic properties of noetherian modules and rings, we introduce a few classes of examples of noetherian rings, which will serve to illustrate and support the later theory. We concentrate particularly on some of the "surrogate" examples outlined in the Prologue, namely, module-finite algebras over commutative rings, skew-Laurent rings, and the corresponding skew polynomial rings twisted by automorphisms. The general theory of skew polynomial rings will be addressed in the following chapter, where we study the Weyl algebras, formal differential operator rings, and other examples from the Prologue.

## • THE NOETHERIAN CONDITION •

We begin with several basic equivalent conditions which are abbreviated by the adjective "noetherian," honoring E. Noether, who first demonstrated the importance and usefulness of these conditions. Recall that a collection $\mathcal{A}$ of subsets of a set $A$ satisfies the *ascending chain condition* (or *ACC*) if there does not exist a properly ascending infinite chain $A_1 \subset A_2 \subset \cdots$ of subsets from $\mathcal{A}$. Recall also that a subset $B \in \mathcal{A}$ is a *maximal element* of $\mathcal{A}$ if there does not exist a subset in $\mathcal{A}$ that properly contains $B$. To emplasize the order-theoretic nature of these considerations, we often use the notation of inequalities ($\leq$, $<$, $\nleq$, etc.) for inclusions among submodules and/or ideals. In particular, if $A$ is a module, the notation $B \leq A$ means that $B$ is a submodule of $A$, and the notation $B < A$ (or $A > B$) means that $B$ is a proper submodule of $A$.

**Proposition 1.1.** *For a module $A$, the following conditions are equivalent:*
(a) *$A$ has the ACC on submodules.*
(b) *Every nonempty family of submodules of $A$ has a maximal element.*
(c) *Every submodule of $A$ is finitely generated.*

*Proof.* (a) $\Longrightarrow$ (b): Suppose that $\mathcal{A}$ is a nonempty family of submodules of $A$ without a maximal element. Choose $A_1 \in \mathcal{A}$. Since $A_1$ is not maximal, there exists $A_2 \in \mathcal{A}$ such that $A_2 > A_1$. Continuing in this manner, we obtain a properly ascending infinite chain $A_1 < A_2 < A_3 < \cdots$ of submodules of $A$, contradicting the ACC.

(b) $\Longrightarrow$ (c): Let $B$ be a submodule of $A$, and let $\mathcal{B}$ be the family of all finitely generated submodules of $B$. Note that $\mathcal{B}$ contains 0 and so is nonempty. By (b), there exists a maximal element $C \in \mathcal{B}$. If $C \neq B$, choose an element $x \in B \setminus C$, and let $C'$ be the submodule of $B$ generated by $C$ and $x$. Then $C' \in \mathcal{B}$ and $C' > C$, contradicting the maximality of $C$. Thus $C = B$, whence $B$ is finitely generated.

(c) $\Longrightarrow$ (a): Let $B_1 \leq B_2 \leq \cdots$ be an ascending chain of submodules of $A$. Let $B$ be the union of the $B_n$. By (c), there exists a finite set $X$ of generators for $B$. Since $X$ is finite, it is contained in some $B_n$, whence $B_n = B$. Thus $B_m = B_n$ for all $m \geq n$, establishing the ACC for submodules of $A$. □

**Definition.** A module $A$ is *noetherian* if and only if the equivalent conditions of Proposition 1.1 are satisfied. As follows from the proof of (b) $\Longrightarrow$ (c), a further equivalent condition is that $A$ have the ACC on *finitely generated* submodules.

For example, any finite dimensional vector space $V$ over a field $k$ is a noetherian $k$-module, since a properly ascending chain of submodules (subspaces) of $V$ cannot contain more than $\dim_k(V) + 1$ terms.

**Definition.** A ring $R$ is *right (left) noetherian* if and only if the right module $R_R$ (left module ${}_RR$) is noetherian. If both conditions hold, $R$ is called a *noetherian ring*.

Rephrasing Proposition 1.1 for the ring itself, we see that a ring $R$ is right (left) noetherian if and only if $R$ has the ACC on right (left) ideals, if and only if all right (left) ideals of $R$ are finitely generated. For example, $\mathbb{Z}$ is a noetherian ring because all its ideals are principal (singly generated). The same is true of a polynomial ring $k[x]$ in one indeterminate over a field $k$.

**Exercise 1A.** (a) Show that the $2 \times 2$ matrices over $\mathbb{Q}$ of the form $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$ with $a \in \mathbb{Z}$ and $b, c \in \mathbb{Q}$ make a ring which is right noetherian but not left noetherian.

(b) Show that any finite direct product of right (left) noetherian rings is right (left) noetherian. □

**Proposition 1.2.** *Let $B$ be a submodule of a module $A$. Then $A$ is noetherian if and only if $B$ and $A/B$ are both noetherian.*

*Proof.* First assume that $A$ is noetherian. Since any ascending chain of submodules of $B$ is also an ascending chain of submodules of $A$, it is immediate that $B$ is noetherian. If $C_1 \leq C_2 \leq \cdots$ is an ascending chain of submodules of $A/B$, each $C_i$ is of the form $A_i/B$ for some submodule $A_i$ of $A$ that contains $B$, and $A_1 \leq A_2 \leq \cdots$. Since $A$ is noetherian, there is some $n$ such that $A_i = A_n$ for all $i \geq n$, and then $C_i = C_n$ for all $i \geq n$. Thus $A/B$ is noetherian.

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Conversely, assume that $B$ and $A/B$ are noetherian, and let $A_1 \leq A_2 \leq \cdots$ be an ascending chain of submodules of $A$. There are ascending chains of submodules

$$A_1 \cap B \leq A_2 \cap B \leq \cdots$$
$$(A_1 + B)/B \leq (A_2 + B)/B) \leq \cdots$$

in $B$ and in $A/B$. Hence, there is some $n$ such that $A_i \cap B = A_n \cap B$ and $(A_i + B)/B = (A_n + B)/B$ for all $i \geq n$, and the latter equation yields $A_i + B = A_n + B$. For all $i \geq n$, we conclude that

$$A_i = A_i \cap (A_i + B) = A_i \cap (A_n + B) = A_n + (A_i \cap B) = A_n + (A_n \cap B) = A_n$$

(using the modular law for the third equality). Therefore $A$ is noetherian. □

In particular, Proposition 1.2 shows that any factor ring of a right noetherian ring is right noetherian. (Note that if $I$ is an ideal of a ring $R$, then the right ideals of $R/I$ are the same as the right $R$-submodules.)

**Corollary 1.3.** *Any finite direct sum of noetherian modules is noetherian.*

*Proof.* It suffices to prove that the direct sum of any two noetherian modules $A_1$ and $A_2$ is noetherian. The module $A = A_1 \oplus A_2$ has a submodule $B = A_1 \oplus 0$ such that $B \cong A_1$ and $A/B \cong A_2$. Then $B$ and $A/B$ are noetherian, whence $A$ is noetherian by Proposition 1.2. □

**Corollary 1.4.** *If $R$ is a right noetherian ring, all finitely generated right $R$-modules are noetherian.*

*Proof.* If $A$ is a finitely generated right $R$-module, then $A \cong F/K$ for some finitely generated free right $R$-module $F$ and some submodule $K \leq F$. Since $F$ is isomorphic to a finite direct sum of copies of the noetherian module $R_R$, it is noetherian by Corollary 1.3. Then, by Proposition 1.2, $A$ must be noetherian. □

**Corollary 1.5.** *Let $S$ be a subring of a ring $R$. If $S$ is right noetherian and $R$ is finitely generated as a right $S$-module, then $R$ is right noetherian.*

*Proof.* By Corollary 1.4, $R$ is noetherian as a right $S$-module. Since all right ideals of $R$ are also right $S$-submodules, the ACC on right ideals follows. □

Using Corollary 1.5, we obtain some easy examples of noncommutative noetherian rings.

**Proposition 1.6.** *If $R$ is a module-finite algebra over a commutative noetherian ring $S$, then $R$ is a noetherian ring.*

*Proof.* The image of $S$ in $R$ is a noetherian subring $S'$ of the center of $R$ such that $R$ is a finitely generated (right or left) $S'$-module. Apply Corollary 1.5. □

For instance, let $S = \mathbb{Z} + \mathbb{Z}\mathbf{i} + \mathbb{Z}\mathbf{j} + \mathbb{Z}\mathbf{k}$, a subring of the division ring $\mathbb{H}$. Since $S$ is a finitely generated module over the noetherian ring $\mathbb{Z}$, Proposition 1.6 shows that $S$ is a noetherian ring. For another example, Proposition 1.6 shows that, for any positive integer $n$, the ring of all $n \times n$ matrices over a commutative noetherian ring is noetherian. This also holds for matrix rings over noncommutative noetherian rings, as follows.

**Definition.** Given a ring $R$ and a positive integer $n$, we use $M_n(R)$ to denote the ring of all $n \times n$ matrices over $R$. The *standard* $n \times n$ *matrix units* in $M_n(R)$ are the matrices $e_{ij}$ (for $i, j = 1, \dots, n$) such that $e_{ij}$ has 1 for the $i, j$-entry and 0 for all other entries.

**Proposition 1.7.** *Let $R$ be a right noetherian ring and $S$ a subring of a matrix ring $M_n(R)$. If $S$ contains the subring*

$$R' = \left\{ \begin{pmatrix} r & 0 & \cdots & 0 \\ 0 & r & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & r \end{pmatrix} \middle| \; r \in R \right\}$$

*of all "scalar matrices," then $S$ is right noetherian. In particular, $M_n(R)$ is a right noetherian ring.*

*Proof.* Clearly $R' \cong R$, whence $R'$ is a right noetherian ring. Observe that $M_n(R)$ is generated as a right $R'$-module by the standard $n \times n$ matrix units. Hence, Corollary 1.4 implies that $M_n(R)$ is a noetherian right $R'$-module. As all right ideals of $S$ are also right $R'$-submodules of $M_n(R)$, we conclude that $S$ is right noetherian. $\square$

## • FORMAL TRIANGULAR MATRIX RINGS •

One way to construct rings to which Corollary 1.5 and Proposition 1.7 apply is to take an upper (or lower) triangular matrix ring over a known ring, or to take a subring of a triangular matrix ring. For instance, if $S$ and $T$ are subrings of a ring $B$, the set $R$ of all matrices of the form $\left(\begin{smallmatrix} s & b \\ 0 & t \end{smallmatrix}\right)$ (for $s \in S$, $b \in B$, $t \in T$) is a subring of $M_2(B)$. (If $S$ and $T$ are right noetherian, and $B_T$ is finitely generated, it follows easily from Corollary 1.5 that $R$ is right noetherian.) Note that $B$ need not be a ring itself in order for $R$ to be a ring – rather, $B$ must be closed under addition, left multiplication by elements of $S$, and right multiplication by elements of $T$. More formally, the symbols $\left(\begin{smallmatrix} s & b \\ 0 & t \end{smallmatrix}\right)$ will form a ring under matrix addition and multiplication provided only that $B$ is simultaneously a left $S$-module and a right $T$-module satisfying an associative law connecting its left and right module structures. We focus on this ring construction because it provides a convenient source for any number of interesting examples. Later, we shall see such left/right modules as $B$ appearing for their own sake in noetherian ring theory.

**Definition.** Let $S$ and $T$ be rings. An $(S,T)$-*bimodule* is an abelian group $B$ equipped with a left $S$-module structure and a right $T$-module structure (both utilizing the given addition) such that $s(bt) = (sb)t$ for all $s \in S$, $b \in B$, $t \in T$. The symbol ${}_SB_T$ is used to denote this situation. An $(S,T)$-*sub-bimodule* of $B$ (or just a *sub-bimodule*, if $S$ and $T$ are clear from the context) is any subgroup of $B$ which is both a left $S$-submodule and a right $T$-submodule. Note that if $C$ is a sub-bimodule of $B$, the factor group $B/C$ is a bimodule in the obvious manner.

For instance, if $S$ is a ring and $T$ is a subring, then $S$ itself (or an ideal of $S$) can be regarded as an $(S,T)$-bimodule (or as a $(T,S)$-bimodule). For another example, if $B$ is a right module over a ring $T$ and $S$ is a subring of $\operatorname{End}_T(B)$, then $B$ is an $(S,T)$-bimodule. Perhaps most importantly, if $I \subseteq J$ are ideals in a ring $S$, then $J/I$ is an $(S,S)$-bimodule. The next exercise shows that in a sense every bimodule appears this way, as an ideal of a formal triangular matrix ring.

**Exercise 1B.** Let ${}_SB_T$ be a bimodule, and write $\begin{pmatrix} S & B \\ 0 & T \end{pmatrix}$ for the abelian group $S \oplus B \oplus T$, where triples $(s,b,t)$ from $S \oplus B \oplus T$ are written as formal $2 \times 2$ matrices $\begin{pmatrix} s & b \\ 0 & t \end{pmatrix}$.

(a) Show that formal matrix addition and multiplication make sense in $\begin{pmatrix} S & B \\ 0 & T \end{pmatrix}$, and that by using those operations $\begin{pmatrix} S & B \\ 0 & T \end{pmatrix}$ becomes a ring.

(b) Show that there is also a ring $\begin{pmatrix} T & 0 \\ B & S \end{pmatrix}$ of formal lower triangular matrices, and that $\begin{pmatrix} T & 0 \\ B & S \end{pmatrix} \cong \begin{pmatrix} S & B \\ 0 & T \end{pmatrix}$.

(c) Observe that the set $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$ of matrices $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ is an ideal of $\begin{pmatrix} S & B \\ 0 & T \end{pmatrix}$, and that, under the obvious abelian group isomorphism of $B$ onto $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$, left $S$-submodules (right $T$-submodules, $(S,T)$-sub-bimodules) of $B$ correspond precisely to left ideals (right ideals, two-sided ideals) of $\begin{pmatrix} S & B \\ 0 & T \end{pmatrix}$ contained in $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$. □

**Definition.** A *formal triangular matrix ring* is any ring of the form $\begin{pmatrix} S & B \\ 0 & T \end{pmatrix}$ or $\begin{pmatrix} T & 0 \\ B & S \end{pmatrix}$ as described in Exercise 1B. By way of abbreviation, we write "let $\begin{pmatrix} S & B \\ 0 & T \end{pmatrix}$ be a formal triangular matrix ring" in place of "let $S$ and $T$ be rings, let $B$ be an $(S,T)$-bimodule, and let $\begin{pmatrix} S & B \\ 0 & T \end{pmatrix}$ be the corresponding formal triangular matrix ring."

Observe that if $S$ and $T$ are subrings of a ring $U$, and $B$ is an $(S,T)$-sub-bimodule of $U$, the formal triangular matrix ring $\begin{pmatrix} S & B \\ 0 & T \end{pmatrix}$ is isomorphic to the subring of $M_2(U)$ consisting of all honest matrices of the form $\left(\begin{smallmatrix} s & b \\ 0 & t \end{smallmatrix}\right)$ with $s \in S$, $b \in B$, $t \in T$.

**Proposition 1.8.** *Let* $R = \left(\begin{smallmatrix} S & B \\ 0 & T \end{smallmatrix}\right)$ *be a formal triangular matrix ring. Then* $R$ *is right noetherian if and only if* $S$ *and* $T$ *are right noetherian and* $B_T$ *is finitely generated. Similarly,* $R$ *is left noetherian if and only if* $S$ *and* $T$ *are left noetherian and* ${}_SB$ *is finitely generated.*

*Proof.* Assume first that $S$ and $T$ are right noetherian and $B_T$ is finitely generated. Observe that the diagonal subring $\left(\begin{smallmatrix} S & 0 \\ 0 & T \end{smallmatrix}\right)$ is isomorphic to $S \times T$ and so is right noetherian. Observe also that if elements $b_1, \dots, b_n$ generate $B$ as a right $T$-module, then the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b_2 \\ 0 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & b_n \\ 0 & 0 \end{pmatrix}$$

generate $R$ as a right $\left(\begin{smallmatrix} S & 0 \\ 0 & T \end{smallmatrix}\right)$-module. Consequently, Corollary 1.5 shows that $R$ is right noetherian.

Conversely, assume that $R$ is right noetherian. Observing that the projection maps $\left(\begin{smallmatrix} s & b \\ 0 & t \end{smallmatrix}\right) \mapsto s$ and $\left(\begin{smallmatrix} s & b \\ 0 & t \end{smallmatrix}\right) \mapsto t$ are ring homomorphisms of $R$ onto $S$ and of $R$ onto $T$, we see that $S$ and $T$ must be right noetherian. Moreover, $\left(\begin{smallmatrix} 0 & B \\ 0 & 0 \end{smallmatrix}\right)$ is a right ideal of $R$ and must have a finite list of generators

$$\begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b_2 \\ 0 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & b_n \\ 0 & 0 \end{pmatrix},$$

from which we infer that the elements $b_1, \dots, b_n$ generate $B_T$.

The left noetherian analog is proved in the same manner. □

For example, it is immediate from Proposition 1.8 that the ring $\left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{smallmatrix}\right)$ is right noetherian but not left noetherian (Exercise 1A(a)).

**Exercise 1C.** Let $R = \left(\begin{smallmatrix} S & B \\ 0 & T \end{smallmatrix}\right)$ be a formal triangular matrix ring. The purpose of this exercise is to give a description of all right $R$-modules in terms of right $S$-modules and $T$-modules.

(a) Let $A$ be a right $S$-module, $C$ a right $T$-module, and $f$ a homomorphism in $\operatorname{Hom}_T(A \otimes_S B, C)$. For $(a, c) \in A \oplus C$ and $\left(\begin{smallmatrix} s & b \\ 0 & t \end{smallmatrix}\right) \in R$, define

$$(a, c)\begin{pmatrix} s & b \\ 0 & t \end{pmatrix} = (as,\ f(a \otimes b) + ct).$$

Show that, using this multiplication rule, $A \oplus C$ is a right $R$-module.

(b) Show that the $R$-module $A \oplus C$ in (a) is finitely generated if and only if $A$ is a finitely generated $S$-module and $C/f(A \otimes_S B)$ is a finitely generated $T$-module.

(c) Show that every right $R$-module is isomorphic to one of the type $A \oplus C$ constructed in (a). □

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**Exercise 1D.** Let ${}_SB_T$ be a bimodule, and form the ring $R = S^{\text{op}} \otimes_{\mathbb{Z}} T$, where $S^{\text{op}}$ denotes the *opposite ring* of $S$. (That is, $S^{\text{op}}$ is the same abelian group as $S$, but with the opposite multiplication: The product of $s_1$ and $s_2$ in $S^{\text{op}}$ is $s_2s_1$.) Show that $B$ can be made into a right $R$-module where $b(s \otimes t) = sbt$ for all $s \in S$, $t \in T$, $b \in B$, and that the right $R$-submodules of $B$ are precisely its $(S,T)$-sub-bimodules. Conversely, show that every right $R$-module can be made into an $(S,T)$-bimodule. □

## • THE HILBERT BASIS THEOREM •

A large class of examples of noetherian rings (particularly, commutative ones) is revealed by this famous theorem. There are several different proofs available; we sketch one that we shall adapt later for skew polynomial rings.

**Theorem 1.9.** [Hilbert's Basis Theorem] *Let $S = R[x]$ be a polynomial ring in one indeterminate. If the coefficient ring $R$ is right (left) noetherian, then so is $S$.*

*Proof.* The two cases are symmetric; let us assume that $R$ is right noetherian and prove that any right ideal $I$ of $S$ is finitely generated. We need only consider the case when $I \neq 0$.

**Step 1**. Let $J$ be the set of leading coefficients of elements of $I$, together with 0. More precisely,

$$J = \{r \in R \mid rx^d + r_{d-1}x^{d-1} + \cdots + r_0 \in I \text{ for some } r_{d-1}, \ldots, r_0 \in R\}.$$

Then check that $J$ is a right ideal of $R$. (Note that if $r, r' \in J$ are leading coefficients of elements $s, s' \in I$ with degrees $d, d'$, then, after replacing $s$ and $s'$ by $sx^{d'}$ and $s'x^d$, we may assume that $s$ and $s'$ have the same degree.)

**Step 2**. Since $R$ is right noetherian, $J$ is finitely generated. Let $r_1, \ldots, r_k$ be a finite list of generators for $J$; we may assume that they are all nonzero. Each $r_i$ occurs as the leading coefficient of a polynomial $p_i \in I$ of some degree $n_i$. Set $n = \max\{n_1, \ldots, n_k\}$ and replace each $p_i$ by $p_i x^{n-n_i}$. Thus, there is no loss of generality in assuming that all the $p_i$ have the same degree $n$.

**Step 3**. Set $N = R + Rx + \cdots + Rx^{n-1} = R + xR + \cdots + x^{n-1}R$, the set of elements of $S$ with degree less than $n$. This is not an ideal of $S$, but it is a left and right $R$-submodule. Viewed as a right $R$-module, $N$ is finitely generated, and so it is noetherian by Corollary 1.4. Now $I \cap N$ is a right $R$-submodule of $N$, and consequently it must be finitely generated. Let $q_1, \ldots, q_t$ be a finite list of right $R$-module generators for $I \cap N$.

**Step 4**. We claim that $p_1, \ldots, p_k, q_1, \ldots, q_t$ generate $I$. Let $I_0$ denote the right ideal of $S$ generated by these polynomials; then $I_0 \subseteq I$ and it remains to show that any polynomial $p \in I$ actually lies in $I_0$. This is easy if $p$ has degree less than $n$, since in that case $p \in I \cap N$ and $p = q_1a_1 + \cdots + q_ta_t$ for some $a_j \in R$.

**Step 5.** Suppose that $p \in I$ has degree $m \geq n$ and that $I_0$ contains all elements of $I$ with degree less than $m$. Let $r$ be the leading coefficient of $p$. Then $r \in J$, and so $r = r_1a_1 + \cdots + r_ka_k$ for some $a_i \in R$. Set $q = (p_1a_1 + \cdots + p_ka_k)x^{m-n}$, an element of $I_0$ with degree $m$ and leading coefficient $r$. Now $p - q$ is an element of $I$ with degree less than $m$. By the induction hypothesis, $p - q \in I_0$, and thus $p \in I_0$.

Therefore $I = I_0$ and we are done. □

It immediately follows that any polynomial ring $R[x_1, \dots, x_n]$ in a finite number of indeterminates over a right (left) noetherian ring $R$ is right (left) noetherian, since we may view $R[x_1, \dots, x_n]$ as a polynomial ring in the single indeterminate $x_n$ with coefficients from the ring $R[x_1, \dots, x_{n-1}]$.

**Corollary 1.10.** *Let $R$ be an algebra over a field $k$. If $R$ is commutative and finitely generated as a $k$-algebra, then $R$ is noetherian.*

*Proof.* Let $x_1, \dots, x_n$ generate $R$ as a $k$-algebra, and let $S = k[y_1, \dots, y_n]$ be a polynomial ring over $k$ in $n$ independent indeterminates. Since $R$ is commutative, there exists a $k$-algebra map $\phi : S \to R$ such that $\phi(y_i) = x_i$ for each $i$, and $\phi$ is surjective because the $x_i$ generate $R$. Hence, $R \cong S/\ker(\phi)$. By the Hilbert Basis Theorem, $S$ is a noetherian ring, and therefore $R$ is noetherian. □

Noncommutative finitely generated algebras need not be noetherian, as the following examples show.

**Exercise 1E.** Let $k$ be a field.

(a) Let $V$ be a countably infinite dimensional vector space over $k$ with a basis $\{v_1, v_2, \dots\}$. Define $s, t \in \operatorname{End}_k(V)$ so that $s(v_i) = v_{i+1}$ for all $i$ while $t(v_i) = v_{i-1}$ for all $i > 1$ and $t(v_1) = 0$, and let $R$ be the $k$-subalgebra of $\operatorname{End}_k(V)$ generated by $s$ and $t$. Show that $R$ is neither right nor left noetherian. [Hint: Define $e_1, e_2, \dots$ in $\operatorname{End}_k(V)$ so that $e_i(v_i) = v_i$ for all $i$ while $e_i(v_j) = 0$ for all $i \neq j$, and show that each $e_i \in R$. Then show that $\sum_i e_iR$ and $\sum_i Re_i$ are not finitely generated.]

(b) If $F$ is the free $k$-algebra on letters $X$ and $Y$, there is a unique $k$-algebra homomorphism $\phi : F \to R$ such that $\phi(X) = s$ and $\phi(Y) = t$. Since $\phi$ is surjective (by definition of $R$), we have $R \cong F/\ker(\phi)$, and so it is clear from part (a) that $F$ cannot be right or left noetherian. Give a direct proof of this fact. [For instance, show that $\sum_i X^iYF$ and $\sum_i FXY^i$ are not finitely generated.] □

**Exercise 1F.** Let $R$ be an algebra over a field $k$, and suppose that $R$ is generated by two elements $x$ and $y$ such that $xy = -yx$. Show that $x^2$ and $y^2$ are in the center of $R$, and that $R$ is a finitely generated module over the subalgebra $S$ generated by $x^2$ and $y^2$. [Hint: Use $1, x, y, xy$ to generate $R$.] Then apply Corollary 1.10 and Proposition 1.6 to conclude that $R$ is noetherian.

Now suppose that, instead of $xy = -yx$, we have $xy = \xi yx$ for some scalar $\xi \in k^{\times}$ which is a root of unity, that is, $\xi^n = 1$ for some positive integer $n$. Modify the steps above to show that $R$ is also noetherian in this case. □

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## • SKEW POLYNOMIAL RINGS TWISTED BY AUTOMORPHISMS •

In the Prologue we saw several examples of rings that look like polynomial rings in one indeterminate but in which the indeterminate does not commute with the coefficients – rather, multiplication by the indeterminate has been "skewed" or "twisted" by means of an automorphism of the coefficient ring, or a derivation, or a combination of such maps. To help the reader get used to constructing and working with such twisted polynomial rings, we begin here by concentrating on the case where the twisting is done by an automorphism. In Chapter 2, we move on to twists by derivations and then to general skew polynomial rings.

Thus, let $R$ be a ring, $\alpha$ an automorphism of $R$, and $x$ an indeterminate. Let $S$ be the set of all formal expressions $a_0 + a_1x + \cdots + a_nx^n$, where $n$ is a nonnegative integer and the $a_i \in R$. It is often convenient to write such an expression as a sum $\sum_i a_ix^i$, leaving it understood that the summation runs over a finite sequence of nonnegative integers $i$, or by thinking of it as an infinite sum in which almost all of the coefficients $a_i$ are zero. We define an addition operation in $S$ in the usual way:

$$\Big(\sum_i a_ix^i\Big) + \Big(\sum_i b_ix^i\Big) = \sum_i (a_i + b_i)x^i.$$

As for multiplication, we would like the coefficients to multiply together as they do in $R$, and we would like the powers of $x$ to multiply following the usual rules for exponents. We take the product of an element $a \in R$ with a power $x^i$ (in that order) to be the single-term sum $ax^i$. It is in a product of the form $x^ia$ that the twist enters. We define $xa$ to be $\alpha(a)x$ and iterate that rule to obtain $x^ia = \alpha^i(a)x$. This leads us to define the following multiplication rule in $S$:

$$\Big(\sum_i a_ix^i\Big)\Big(\sum_j b_jx^j\Big) = \sum_{i,j} a_i\alpha^i(b_j)x^{i+j} = \sum_k \Big(\sum_{i+j=k} a_i\alpha^i(b_j)\Big)x^k.$$

**Exercise 1G.** Verify that the set $S$ together with the operations defined above is a ring, and that when $R$ is identified with the set of elements of $S$ involving no positive powers of $x$, it becomes a subring of $S$. □

**Exercise 1H.** Here is a more formal description of $S$, in which the symbol $x$ does not make an a priori appearance.

Let $\overline{S}$ denote the set of those infinite sequences $a = (a_0, a_1, a_2, \dots)$ of elements of $R$ in which $a_i = 0$ for all but finitely many indices $i$. For any $a, b \in \overline{S}$, define $a+b$ and $ab$ to be the sequences in $\overline{S}$ with entries

$$(a+b)_i = a_i + b_i \qquad\qquad (ab)_k = \sum_{i+j=k} a_i \alpha^i(b_j)$$

for all $i$ and $k$. Show that $\overline{S}$ with these operations is a ring, and that $\overline{S} \cong S$ via the rule $a \mapsto \sum_i a_i x^i$. This isomorphism makes it clear that $x$ is just a name for a particular special element of $S$, corresponding to the sequence $(0, 1, 0, 0, 0, \dots)$ in $\overline{S}$. $\square$

We have glossed over an important point in our discussion of $S$ – the question of when two formal expressions define the same element of $S$. Namely, we have taken it as understood that two elements of $S$ are the same only if their coefficients are the same, that is, $\sum_i a_i x^i = \sum_i b_i x^i$ if and only if $a_i = b_i$ for all $i$. Missing coefficients are understood to be zero: In case the equation concerns finite sums and an index $i$ occurs in the first sum but not in the second, equality of coefficients means that $a_i = 0$. Using the language of linear algebra, we can thus say that the elements $1, x, x^2, \dots$ in $S$ are linearly independent over $R$. Since every element of $S$ is a linear combination of these powers, $S$ is thus a free left $R$-module with the powers of $x$ forming a basis. This leads us to the following definition.

**Definition.** Let $R$ be a ring and $\alpha$ an automorphism of $R$. We write

$$S = R[x; \alpha]$$

(where $S$ and $x$ may or may not already occur in the discussion) to mean that

(a) $S$ is a ring, containing $R$ as a subring;
(b) $x$ is an element of $S$;
(c) $S$ is a free left $R$-module with basis $\{1, x, x^2, \dots\}$;
(d) $xr = \alpha(r)x$ for all $r \in R$.

Thus, the expression $S = R[x; \alpha]$ can be used either to introduce a new ring $S$ (constructed as above) or to say that a given ring $S$ and element $x$ satisfy conditions (a)–(d). Whenever $S = R[x; \alpha]$, we say that $S$ is a *skew polynomial ring over $R$.*

In many algebra texts, rings of polynomials are introduced as specific rings resulting from special constructions. Note that the definition above is of a different type, since $R[x; \alpha]$ is defined to be any ring extension of $R$ satisfying certain properties, rather than as any specific ring (although some construction is needed to guarantee that such skew polynomial rings exist). In