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978-0-521-54420-7 - Elements of the Representation Theory of Associative Algebras,
Volume 2: Tubes and Concealed Algebras of Euclidean type

Daniel Simson and Andrzej Skowronski

Excerpt

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Chapter X

Tubes

In Chapter VIII of Volume 1, we have started to study the Auslander–Reiten quiver $\Gamma(\text{mod } A)$ of any hereditary K -algebra A of Euclidean type, that is, the path algebra $A = KQ$ of an acyclic quiver Q whose underlying graph \bar{Q} is one of the Euclidean diagrams \tilde{A}_m , with $m \geq 1$, \tilde{D}_m , with $m \geq 4$, \tilde{E}_6 , \tilde{E}_7 , and \tilde{E}_8 . We recall that any such an algebra A is representation-infinite.

We have shown in (VIII.2.3) that the quiver $\Gamma(\text{mod } A)$ contains a unique postprojective component $\mathcal{P}(A)$ containing all the indecomposable projective A -modules, a unique preinjective component $\mathcal{Q}(A)$ containing all the indecomposable injective A -modules, and the family $\mathcal{R}(A)$ of the remaining components being called regular (see (VIII.2.12)). This means that $\Gamma(\text{mod } A)$ has the disjoint union form

$$\Gamma(\text{mod } A) = \mathcal{P}(A) \cup \mathcal{R}(A) \cup \mathcal{Q}(A).$$

The indecomposable modules in $\mathcal{R}(A)$ are called regular. We have shown in (VIII.4.5) that there is a similar structure of $\Gamma(\text{mod } B)$, for any concealed algebra B of Euclidean type, that is, the endomorphism algebra

$$B = \text{End } T_A$$

of a postprojective tilting module T_A over a hereditary algebra $A = KQ$ of Euclidean type. The algebra B is representation-infinite.

The objective of Chapters XI–XIII is to describe the structure of regular components of the Auslander–Reiten quiver $\Gamma(\text{mod } B)$ of any concealed algebra B of Euclidean type.

We introduce in this chapter a special type of a translation quiver, which we call a stable tube. The main aim of Section 1 is to describe special properties of irreducible morphisms between indecomposable modules in stable tubes of the Auslander–Reiten quiver $\Gamma(\text{mod } B)$ of an algebra B and their compositions with arbitrary homomorphisms in the module category $\text{mod } B$. In particular, some relevant properties of the radical rad_B and the infinite radical rad_B^∞ of the category $\text{mod } B$ of finite dimensional right B -modules are described.

In Section 2, we introduce the important concept of a standard component and we prove Ringel’s handy criterion on the existence of a standard self-hereditary stable tube in the Auslander–Reiten quiver $\Gamma(\text{mod } B)$ of any algebra B . By applying the criterion, we show in Chapter XI that the regular components of any (representation-infinite) concealed algebra B of Euclidean type are self-hereditary standard stable tubes.

In Section 3, we introduce the concept of a generalised standard component of $\Gamma(\text{mod } B)$, invoking the infinite radical rad_B^∞ of the category $\text{mod } B$, and exhibit basic examples of generalised standard components. The main result of Section 4 is a characterisation of (generalised) standard stable tubes obtained by Skowroński in [246], [247], and [254]. It asserts that, for a stable tube \mathcal{T} in the Auslander–Reiten quiver $\Gamma(\text{mod } B)$ of any algebra B , the following three statements are equivalent:

- \mathcal{T} is a standard stable tube,
- the mouth of \mathcal{T} consists of pairwise orthogonal bricks, and
- \mathcal{T} is a generalised standard stable tube.

It is also shown that $\text{pd } X = 1$ and $\text{id } X = 1$, for any indecomposable B -module lying in a faithful generalised standard stable tube \mathcal{T} of $\Gamma(\text{mod } B)$.

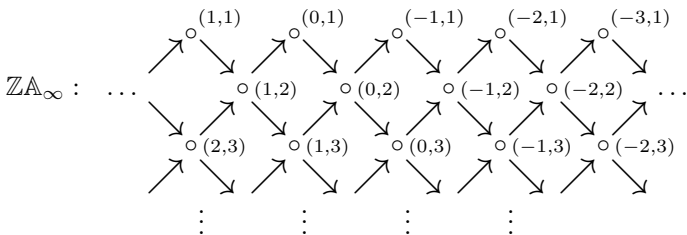
Throughout, we assume that K is an algebraically closed field, and by an algebra we mean a finite dimensional K -algebra. Given a finite quiver $Q = (Q_0, Q_1)$, we denote by KQ the path K -algebra of Q . We recall that the dimension $\dim_K KQ$ of KQ is finite if and only if the quiver Q is **acyclic**, that is, there is no oriented cycle in Q , see Chapters II and III.

X.1. Stable tubes

We have defined in (VIII.1.1) the translation quiver $\mathbb{Z}\Sigma$, for Σ being a connected and acyclic quiver. Thus, letting Σ be the infinite quiver

$$\mathbb{A}_\infty : \circ_1 \longrightarrow \circ_2 \longrightarrow \circ_3 \longrightarrow \circ_4 \longrightarrow \dots \longrightarrow \circ_m \longrightarrow \circ_{m+1} \longrightarrow \dots$$

we obtain the **infinite translation quiver**



where $\tau(n, i) = (n + 1, i)$, for $n \in \mathbb{Z}$ and $i \geq 1$. Thus, by definition, τ is an automorphism of $\mathbb{Z}\mathbb{A}_\infty$, and hence so is any power τ^r of τ (with $r \in \mathbb{Z}$). For a fixed $r \geq 1$, let (τ^r) denote the infinite cyclic group of automorphisms of $\mathbb{Z}\mathbb{A}_\infty$ generated by τ^r , and let $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ denote the orbit space of $\mathbb{Z}\mathbb{A}_\infty$ under the action of (τ^r) . That is, $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ is the translation quiver obtained from $\mathbb{Z}\mathbb{A}_\infty$ by identifying each point (n, i) of $\mathbb{Z}\mathbb{A}_\infty$ with the point $\tau^r(n, i) = (n + r, i)$, and each arrow $\alpha : x \rightarrow y$ in $\mathbb{Z}\mathbb{A}_\infty$ with the arrow $\tau^r \alpha : \tau^r x \rightarrow \tau^r y$. We are thus led to the following definition.

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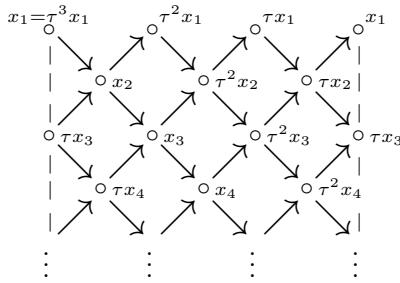
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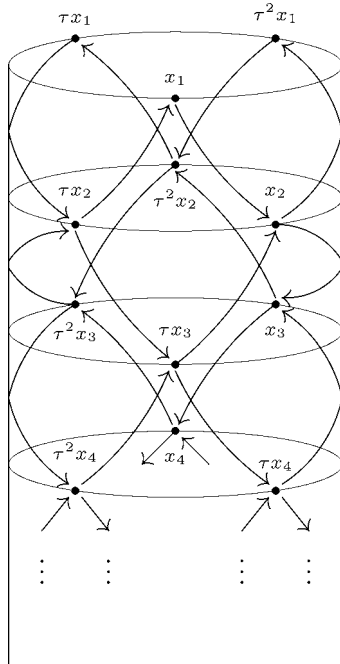
1.1. Definition. Let (\mathcal{T}, τ) be a translation quiver.

- (a) (\mathcal{T}, τ) is defined to be a **stable tube of rank** $r = r_{\mathcal{T}} \geq 1$ if there is an isomorphism of translation quivers $\mathcal{T} \cong \mathbb{Z}\mathbb{A}_{\infty}/(\tau^r)$.
- (b) A stable tube of rank $r = 1$ is defined to be a **homogeneous tube**.
- (c) Let (\mathcal{T}, τ) be a stable tube of rank $r \geq 1$. A sequence (x_1, \dots, x_r) of points of \mathcal{T} is said to be a **τ -cycle** if $\tau x_1 = x_r, \tau x_2 = x_1, \dots, \tau x_r = x_{r-1}$.

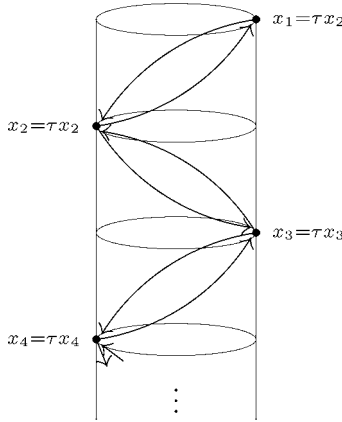
For example, a stable tube of rank 3 is obtained from the quiver



by identifying along the vertical dotted lines, thus giving the following



Similarly, a homogeneous tube has the following shape



We observe that the translation τ still acts as an automorphism over a stable tube of rank r (that is the reason why such tubes are called stable), and that τ^r acts as the identity. The latter fact is expressed by saying that any point of $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ is τ -periodic of period r .

We recall from Section IX.2 that a path $x_0 \rightarrow \dots \rightarrow x_t$ in a translation quiver is called **sectional** if $\tau x_i \not\cong x_{i-2}$, for all $i \in \{2, \dots, t\}$.

The following two definitions are of importance in the theory.

1.2. Definition. Let (\mathcal{T}, τ) be a stable tube.

- (a) The set of all points in \mathcal{T} having exactly one immediate predecessor (or, equivalently, exactly one immediate successor) is called the **mouth** of \mathcal{T} .
- (b) Given a point x lying on the mouth of the stable tube \mathcal{T} , a **ray** starting at x is defined to be a unique infinite sectional path

$$x = x[1] \rightarrow x[2] \rightarrow x[3] \rightarrow x[4] \rightarrow \dots \rightarrow x[m] \rightarrow \dots$$
 in the tube \mathcal{T} .
- (c) Given a point x lying on the mouth of the stable tube \mathcal{T} , a **coray** ending with x is defined to be a unique infinite sectional path

$$\dots \rightarrow [m]x \rightarrow \dots \rightarrow [4]x \rightarrow [3]x \rightarrow [2]x \rightarrow [1]x = x$$
 in the tube \mathcal{T} .

To see that the definition is correct, we note that, for each point x lying on the mouth of a stable tube \mathcal{T} , there exists a unique arrow starting at x and a unique arrow ending at x . Because an arbitrary point in \mathcal{T} is the source (and the target) of at most two arrows, this implies the existence of a unique infinite sectional path in \mathcal{T} starting at x , and a unique infinite sectional path in \mathcal{T} ending with x .

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1.3. Definition. Let A be an algebra and \mathcal{C} be a component of the Auslander–Reiten quiver $\Gamma(\text{mod } A)$ of A .

- (a) A **ray point** of \mathcal{C} is defined to be a point X in \mathcal{C} such that there exists an infinite sectional path in \mathcal{C}

$$X = X[1] \longrightarrow X[2] \longrightarrow X[3] \longrightarrow X[4] \longrightarrow \dots \longrightarrow X[m] \longrightarrow \dots$$

starting at X and containing all sectional paths starting at X . The corresponding A -module X is called a **ray module**. The unique infinite sectional path starting at X is called the **ray** starting at X .

- (b) A **coray point** of \mathcal{C} is defined to be a point X in \mathcal{C} such that there exists an infinite sectional path in \mathcal{C}

$$\dots \longrightarrow [m]X \longrightarrow \dots \longrightarrow [4]X \longrightarrow [3]X \longrightarrow [2]X \longrightarrow [1]X = X$$

ending with X and containing all sectional paths ending with X . The corresponding A -module X is called the **coray module**. The unique infinite sectional path ending with X is called the **coray** ending with X .

A **ray point of a stable tube** \mathcal{T} and a **coray point of a stable tube** \mathcal{T} are defined analogously.

It is easy to see that if (\mathcal{T}, τ) is a stable tube and x is a point x in \mathcal{T} , then the following three statements are equivalent:

- x is a ray point of the tube \mathcal{T} ,
- x is a coray point of \mathcal{T} , and
- x lies on the mouth of the tube \mathcal{T} .

Now we collect basic facts on the structure of any stable tube of $\Gamma(\text{mod } A)$.

1.4. Lemma. *Let A be an algebra, and \mathcal{T} a stable tube of rank $r = r_{\mathcal{T}} \geq 1$ of the Auslander–Reiten quiver $\Gamma(\text{mod } A)$ of A . Assume that (X_1, \dots, X_r) is a τ_A -**cycle** of (indecomposable) mouth modules of mouth A -modules of the tube \mathcal{T} , that is, the modules X_1, \dots, X_r lie on the mouth of \mathcal{T} and satisfy $\tau_A X_1 \cong X_r, \tau_A X_2 = X_1, \dots, \tau_A X_r = X_{r-1}$.*

- (a) *For each $i \in \{1, \dots, r\}$, there exists a unique ray*

$$(r_i) \quad X_i = X_i[1] \longrightarrow X_i[2] \longrightarrow X_i[3] \longrightarrow \dots \longrightarrow X_i[m] \longrightarrow X_i[m+1] \longrightarrow \dots$$

in \mathcal{T} starting at X_i , and a unique coray

$$(c_i) \quad \dots \longrightarrow [m+1]X_i \longrightarrow [m]X_i \longrightarrow \dots \longrightarrow [3]X_i \longrightarrow [2]X_i \longrightarrow [1]X_i = X_i$$

in \mathcal{T} ending with X_i .

- (b) *Every indecomposable A -module M in \mathcal{T} is of the form $M \cong X_i[m]$, for some $i \in \{1, \dots, r\}$ and $m \geq 1$.*

- (c) Every indecomposable A -module M in \mathcal{T} is of the form $M \cong [m]X_s$, for some $s \in \{1, \dots, r\}$ and $m \geq 1$.
- (d) $[m]X_s \cong X_{s-m+1}[m]$ and $X_s[m] \cong [m]X_{s+m-1}$, for each $s \in \{1, \dots, r\}$ and $m \geq 1$, where $s - m + 1$ is reduced modulo $r - 1$ if $s - m + 1 \leq 0$ or $s - m + 1 \geq r$.
- (e) Under the isomorphisms of A -modules

$$[1]X_i \cong X_i[1], [2]X_i \cong X_{i-1}[2], \dots, [m]X_i \cong X_{i+m-1}[m], \dots$$

the coray (\mathbf{c}_i) has the form

$$(\mathbf{c}_i) \dots \rightarrow X_{i-m}[m+1] \rightarrow X_{i-m+1}[m] \rightarrow \dots \rightarrow X_{i-1}[2] \rightarrow X_i[1] = X_i.$$

- (f) For any $i \in \{1, \dots, r\}$ and $m \geq 1$, there exists an almost split sequence

$$0 \rightarrow X_i[m] \xrightarrow{\begin{bmatrix} f_{i,m+1} \\ g_{i,m} \end{bmatrix}} X_i[m+1] \oplus X_{i+1}[m-1] \xrightarrow{[g_{i,m+1} \ f_{i+1,m}]} X_{i+1}[m] \rightarrow 0$$

in $\text{mod } A$, where we set $X_i[0] = 0$ and $X_{i+kr}[j] = X_i[j]$, for all $i \in \{1, \dots, r\}$, $j \geq 1$, and $k \in \mathbb{Z}$.

Proof. Assume that (X_1, \dots, X_r) is a τ_A -cycle of mouth modules of the tube \mathcal{T} of rank $r \geq 1$. Then X_1, \dots, X_r are indecomposable, lie on the mouth of the tube \mathcal{T} , and there are isomorphisms $\tau_A X_1 \cong X_r, \tau_A X_2 = X_1, \dots, \tau_A X_r = X_{r-1}$. Because the tube \mathcal{T} is stable then there is a surjective morphism $f : \mathbb{Z}\mathbb{A}_\infty \longrightarrow \mathcal{T}$ of translation quivers such that $f(-1, 1) = X_1, f(-2, 1) = X_2, \dots, f(-r, 1) = X_r$ and the induced morphism $\tilde{f} : \mathbb{Z}\mathbb{A}_\infty / (\tau^r) \xrightarrow{\cong} \mathcal{T}$ is an isomorphism. It is clear that the conditions (a), (b), and (c) are satisfied in the translation quiver $(\mathbb{Z}\mathbb{A}_\infty, \tau)$. Hence we easily conclude that (a), (b), and (c) hold in \mathcal{T} , if we set $X_i[m] = f(-i, m)$ and $[m]X_i = f(-i+m-1, m)$.

The statement (d) follows from (a), (b), and (c) by an easy induction on $m \geq 1$.

Now we prove (e). It follows from (d) that, given $i \in \{1, \dots, r\}$ and $m \geq 1$, there are isomorphisms $X_{i-m}[m+1] \cong [m+1]X_i$ and $X_{i-m+1}[m] \cong [m]X_i$. Hence, the following arrow in the coray (\mathbf{c}_i)

$$X_{i-m}[m+1] \cong [m+1]X_i \longrightarrow [m]X_i \cong X_{i-m+1}[m]$$

corresponds to an irreducible morphism $X_{i-m}[m+1] \rightarrow X_{i-m+1}[m]$ in $\text{mod } A$. To prove (f), we note that in view of the shape of the stable tube \mathcal{T} , each of its vertices is a source of at most two arrows, and the arrows correspond to some irreducible morphisms in $\text{mod } A$; thus yield a required almost split sequence. The proof of the lemma is then complete. \square

In the remaining part of this section we investigate properties of irreducible morphisms between indecomposable modules in a stable tube \mathcal{T} of

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the Auslander–Reiten quiver $\Gamma(\text{mod } A)$ of an algebra A . The investigation needs some new concepts and preliminary results.

Assume that A is an arbitrary algebra. We recall from Section A.3 of Volume 1 that the **radical** $\text{rad}_A = \text{rad}(\text{mod } A)$ of the category $\text{mod } A$ is the two-sided ideal of $\text{mod } A$ defined by the formula

$$\text{rad}_A(X, Y) = \{h \in \text{Hom}_A(X, Y); 1_X - gh \text{ is invertible, for any } g : Y \rightarrow X\},$$

for each pair of modules X and Y in $\text{mod } A$. If the A -modules X and Y are indecomposable then $\text{rad}_A(X, Y)$ consists of all non-isomorphisms $h : X \rightarrow Y$ in $\text{mod } A$, see (A.3.4) and (A.3.5) of Volume 1. In particular, $\text{rad}_A(X, X)$ is just the radical of the local algebra $\text{End } X$, for any indecomposable A -module X .

In other words, the radical rad_A of the category $\text{mod } A$ is the two-sided ideal of $\text{mod } A$ generated by all non-isomorphisms between indecomposable A -modules.

Given $m \geq 1$, the m th power $\text{rad}_A^m \subseteq \text{rad}_A$ of rad_A is the two-sided ideal of $\text{mod } A$ such that, for A -modules X and Y , $\text{rad}_A^m(X, Y)$ is the subspace of $\text{rad}_A(X, Y)$ consisting of all finite sums of composite homomorphisms of the form

$$X = X_0 \xrightarrow{h_1} X_1 \xrightarrow{h_2} X_2 \rightarrow \dots \rightarrow X_{m-1} \xrightarrow{h_m} X_m = Y,$$

where $h_j \in \text{rad}_A(X_{j-1}, X_j)$, for any $j \in \{1, 2, \dots, m-1, m\}$. For $m = 0$, we set $\text{rad}_A^m(X, Y) = \text{Hom}_A(X, Y)$. The intersection

$$\text{rad}_A^\infty = \bigcap_{m=1}^{\infty} \text{rad}_A^m$$

of all powers rad_A^m of rad_A is called the **infinite radical** of $\text{mod } A$.

We recall from (IV.1.6) that a homomorphism $h : X \rightarrow Y$ between two indecomposable modules X and Y in $\text{mod } A$ is an irreducible morphism if and only if $h \in \text{rad}_A(X, Y) \setminus \text{rad}_A^2(X, Y)$. It follows that the radical rad_A of the category $\text{mod } A$ is the two-sided ideal of $\text{mod } A$ generated by the irreducible morphisms in $\text{mod } A$, as a left ideal and as a right ideal.

Throughout, we need the following two preliminary lemmata.

1.5. Lemma. *Let A be an algebra, and M, N be a pair of modules in $\text{mod } A$.*

- (a) *There exists an integer $m \geq 0$ such that $\text{rad}_A^\infty(M, N) = \text{rad}_A^m(M, N)$.*
- (b) *$\text{rad}_A^\infty(M, N) = \text{rad}_A(M, N)$, if the modules M and N are indecomposable and lie in two different components \mathcal{C}_1 and \mathcal{C}_2 of the Auslander–Reiten quiver $\Gamma(\text{mod } A)$ of A .*

Proof. (a) Because the vector space $\text{Hom}_A(M, N)$ is finite dimensional, the descending chain

$\text{Hom}_A(M, N) \supseteq \text{rad}_A(M, N) \supseteq \text{rad}_A^2(M, N) \supseteq \dots \supseteq \text{rad}_A^m(M, N) \supseteq \dots$ terminates, that is, $\text{rad}_A^m(M, N) = \text{rad}_A^{m+1}(M, N) = \text{rad}_A^{m+2}(M, N) = \dots$, for some $m \geq 0$. It follows that $\text{rad}_A^\infty(M, N) = \text{rad}_A^m(M, N)$.

The statement (b) is a consequence of (IV.5.1). □

1.6. Lemma. *Let A be an algebra, M an A -module, and*

$$0 \longrightarrow Z \xrightarrow{\begin{bmatrix} g \\ g' \end{bmatrix}} X \oplus X' \xrightarrow{[f, f']} Y \longrightarrow 0$$

an almost split sequence in $\text{mod } A$, where X, X', Y , and Z are indecomposable modules.

- (a) *Let $\ell \geq 1$ be an integer and $h : M \rightarrow X$ be a homomorphism in $\text{mod } A$ such that $h \notin \text{rad}_A^\ell(M, X)$ and $fh \in \text{rad}_A^{\ell+1}(M, Y)$. Then there exists a homomorphism $h' : M \rightarrow Z$ in $\text{mod } A$ such that $h' \notin \text{rad}_A^{\ell-1}(M, Z)$ and $g'h' \in \text{rad}_A^\ell(M, X')$.*
- (b) *Let $\ell \geq 1$ be an integer and $t : X \rightarrow M$ be a homomorphism in $\text{mod } A$ such that $t \notin \text{rad}_A^\ell(X, M)$ and $tg \in \text{rad}_A^{\ell+1}(Z, M)$. Then there exists a homomorphism $t' : Y \rightarrow M$ in $\text{mod } A$ such that $t' \notin \text{rad}_A^{\ell-1}(Y, M)$ and $t'f' \in \text{rad}_A^\ell(X', M)$.*

Proof. We only prove the statement (a), because the proof of (b) is similar.

Assume that $h \notin \text{rad}_A^\ell(M, X)$ and $fh \in \text{rad}_A^{\ell+1}(M, Y)$. Then there exists an A -module N and two homomorphisms $w \in \text{rad}_A^\ell(M, N)$ and $v \in \text{rad}_A(N, Y)$ such that $fh = vw$. Then v is not a retraction and, hence, there exists a homomorphism $\begin{bmatrix} u \\ u' \end{bmatrix} : N \rightarrow X \oplus X'$ such that $v = [f, f'] \cdot \begin{bmatrix} u \\ u' \end{bmatrix} = fu + f'u'$. Consider the homomorphisms

$$\begin{bmatrix} uw-h \\ u'w \end{bmatrix} : M \rightarrow X \oplus X'$$

and note that

$$[f, f'] \cdot \begin{bmatrix} uw-h \\ u'w \end{bmatrix} = f uw - fh + f'u'w = (fu + f'u')w - fh = vw - fh = 0.$$

It follows that there exists a homomorphism $h' : N \rightarrow Z$ such that

$$\begin{bmatrix} uw-h \\ u'w \end{bmatrix} = \begin{bmatrix} g \\ g' \end{bmatrix} \cdot h',$$

and therefore $g'h' = u'w \in \text{rad}_A^\ell(M, X')$, because $\begin{bmatrix} g \\ g' \end{bmatrix}$ is the kernel of $[f, f']$. Note also that $h' \notin \text{rad}_A^{\ell-1}(M, Z)$, because otherwise we get the contradiction $h = uw - gh' \in \text{rad}_A^\ell(M, X)$. This finishes the proof. □

1.7. Proposition. *Let A be an algebra, \mathcal{T} a stable tube of rank $r = r_{\mathcal{T}} \geq 1$ of the Auslander–Reiten quiver $\Gamma(\text{mod } A)$ of A , and (X_1, \dots, X_r) a τ_A -cycle of mouth modules of the tube \mathcal{T} . In the notation of (1.4), let (τ_i)*

and (\mathfrak{c}_i) be the ray starting at X_i and the coray ending with X_i in \mathcal{T} , for each $i \in \{1, \dots, r\}$.

- (a) For each $i \in \{1, \dots, r\}$ and $m \geq 2$, any irreducible morphism $f_{i,m} : X_i[m-1] \rightarrow X_i[m]$ corresponding to the arrow $X_i[m-1] \rightarrow X_i[m]$ in the ray (\mathfrak{r}_i) of \mathcal{T} starting at X_i is a monomorphism.
- (b) For each $i \in \{1, \dots, r\}$ and $m \geq 2$, any irreducible morphism $g_{i,m} : [m]X_i \rightarrow [m-1]X_i$ corresponding to the arrow $[m]X_i \rightarrow [m-1]X_i$ in the coray (\mathfrak{c}_i) of \mathcal{T} ending with X_i is an epimorphism.
- (c) For each $i \in \{1, \dots, r\}$ and $m \geq 2$, there exist irreducible morphisms $v_{i,m} : X_i[m-1] \rightarrow X_i[m]$ and $q_{i,m} : X_i[m] \rightarrow X_{i+1}[m-1]$ in $\text{mod } A$ such that
 - (c1) $q_{i,2}v_{i,2} \in \text{rad}_A^3(X_i[1], X_{i+1}[1])$, and
 - (c2) $v_{i+1,m}q_{i,m} + q_{i,m+1}v_{i,m+1} \in \text{rad}_A^3(X_i[m], X_{i+1}[m])$.

Proof. Assume that \mathcal{T} is a stable tube of rank $r = r_{\mathcal{T}} \geq 1$ of $\Gamma(\text{mod } A)$ and (X_1, \dots, X_r) is a τ_A -cycle of mouth modules of the tube \mathcal{T} , that is, the modules X_1, \dots, X_r are indecomposable, lie on the mouth of the tube \mathcal{T} , and there are isomorphisms $\tau_A X_1 \cong X_r, \tau_A X_2 = X_1, \dots, \tau_A X_r = X_{r-1}$. Then (1.4) applies, and we freely use the notation introduced there. Clearly, for each $i \in \{1, \dots, r\}$ and $m \geq 2$, the arrow $X_i[m-1] \rightarrow X_i[m]$ in the ray (\mathfrak{r}_i) of \mathcal{T} starting from X_i corresponds to an irreducible morphism $f_{i,m} : X_i[m-1] \rightarrow X_i[m]$ in $\text{mod } A$. By (1.4)(d), there is an isomorphism $[m]X_s \cong X_{s-m+1}[m]$, for each $s \in \{1, \dots, r\}$ and $m \geq 1$, where $s - m + 1$ is reduced modulo $r - 1$ if $s - m + 1 \leq 0$ or $s - m + 1 \geq r$. It follows that, given $i \in \{1, \dots, r\}$ and $m \geq 1$, there are isomorphisms $X_i[m] \cong [m]X_{i+m-1}$ and $X_{i+1}[m-1] \cong [m-1]X_{i+m-1}$. Hence, the arrow

$$X_i[m] \cong [m]X_{i+m-1} \rightarrow [m-1]X_{i+m-1} \cong X_{i+1}[m-1]$$

in the coray (\mathfrak{c}_i) of \mathcal{T} ending with X_i corresponds to an irreducible morphism $g_{i,m} : X_i[m] \rightarrow X_{i+1}[m-1]$ in $\text{mod } A$. In view of the shape of the stable tube \mathcal{T} , each of its vertices is a source of at most two arrows. It follows that, for any $i \in \{1, \dots, r\}$ and $m \geq 1$, there exists an almost split sequence

$$0 \rightarrow X_i[m] \xrightarrow{\begin{bmatrix} f_{i,m+1} \\ g_{i,m} \end{bmatrix}} X_i[m+1] \oplus X_{i+1}[m-1] \xrightarrow{[g_{i,m+1} \ f_{i+1,m}]} X_{i+1}[m] \rightarrow 0$$

in $\text{mod } A$, where we set $X_i[0] = 0$ and $X_{i+kr}[j] = X_i[j]$, for all $i \in \{1, \dots, r\}$, $j \geq 1$, and $k \in \mathbb{Z}$. Counting the dimensions we get

$$\dim_K X_i[m] + \dim_K X_{i+1}[m] = \dim_K X_i[m+1] + \dim_K X_i[m-1].$$

Hence, we conclude the inequalities $\dim_K X_i[1] < \dim_K X_i[2] > \dim_K X_{i+1}[1]$, and an easy induction on $m \geq 1$ shows that

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$$\dim_K X_i[m] < \dim_K X_i[m+1] > \dim_K X_{i+1}[m],$$

for all $i \in \{1, \dots, r\}$ and $m \geq 1$. Hence we easily conclude that any irreducible morphism $f_{i,m} : X_i[m-1] \rightarrow X_i[m]$ is a monomorphism and any irreducible morphism $g_{i,m} : [m]X_i \rightarrow [m-1]E_i$ is an epimorphism, because we know from (IV.1.4) that every irreducible morphism in $\text{mod } A$ is a proper monomorphism or a proper epimorphism. This finishes the proof of (a) and (b).

Now we prove (c). First we observe that if $f, g : X \rightarrow Y$ are irreducible morphisms in $\text{mod } A$ and X, Y are indecomposable modules lying in the tube \mathcal{T} then

$$f + \text{rad}_A^2(X, Y) = \lambda \cdot g + \text{rad}_A^2(X, Y),$$

for some $\lambda \in K \setminus \{0\}$, because

$$\dim_K \text{rad}_A(X, Y) / \text{rad}_A^2(X, Y) = 1,$$

by (IV.1.6), (IV.4.6), and the definition of a stable tube. For each $i \in \{1, \dots, r\}$, we choose some irreducible morphisms

$$v_{i,2} : X_i[1] \longrightarrow X_i[2] \quad \text{and} \quad q_{i,2} : X_i[2] \longrightarrow X_{i+1}[1]$$

in $\text{mod } A$, and consider an almost split sequence

$$0 \longrightarrow X_i[1] \xrightarrow{u_{i,2}} X_i[2] \xrightarrow{p_{i,2}} X_{i+1}[1] \longrightarrow 0,$$

in $\text{mod } A$. By our earlier observations, there exist scalars $\lambda_i, \mu_i \in K \setminus \{0\}$ and homomorphisms $w_{i,2} \in \text{rad}_A^2(X_i[1], X_i[2])$ and $t_{i,2} \in \text{rad}_A^2(X_i[2], X_{i+1}[1])$ such that $v_{i,2} = \lambda_i u_{i,2} + w_{i,2}$ and $q_{i,2} = \mu_i p_{i,2} + t_{i,2}$. Then we get the equalities

$$\begin{aligned} q_{i,2} v_{i,2} &= (\mu_i p_{i,2} + t_{i,2})(\lambda_i u_{i,2} + w_{i,2}) \\ &= \mu_i \lambda_i p_{i,2} u_{i,2} + (\mu_i p_{i,2} w_{i,2} + \lambda_i t_{i,2} u_{i,2} + t_{i,2} w_{i,2}) \\ &= \mu_i p_{i,2} w_{i,2} + \lambda_i t_{i,2} u_{i,2} + t_{i,2} w_{i,2} \in \text{rad}_A^3(X_i[1], X_{i+1}[1]), \end{aligned}$$

and (c1) follows.

Assume that $s \geq 2$ and, for all $i \in \{1, \dots, r\}$ and $k \in \{2, \dots, s\}$, there exist irreducible morphisms

$$v_{i,k} : X_i[k-1] \longrightarrow X_i[k] \quad \text{and} \quad q_{i,k} : X_i[k] \longrightarrow X_{i+1}[k-1]$$

in $\text{mod } A$ satisfying (c1) and (c2), for $k \in \{2, \dots, s-1\}$. Fix $i \in \{1, \dots, r\}$ and consider an almost split sequence

$$0 \longrightarrow X_i[s] \xrightarrow{\begin{bmatrix} u_{i,s+1} \\ p_{i,s} \end{bmatrix}} X_i[s+1] \oplus X_{i+1}[s-1] \xrightarrow{[p_{i,s+1} \quad u_{i+1,s}]} X_{i+1}[s] \longrightarrow 0$$

in $\text{mod } A$. Then there exist scalars $\lambda_{i+1}^{(s)}, \mu_i^{(s)} \in K \setminus \{0\}$ and homomorphisms $w_{i+1,s} \in \text{rad}_A^2(X_{i+1}[s-1], X_{i+1}[s])$ and $t_{i,s} \in \text{rad}_A^2(X_i[s], X_{i+1}[s-1])$ such that $v_{i+1,s} = \lambda_{i+1}^{(s)} u_{i+1,s} + w_{i+1,s}$ and $q_{i,s} = \mu_i^{(s)} p_{i,s} + t_{i,s}$.