

*Chapter I*Fourier Series on \mathbb{T}

We denote by \mathbb{R} the additive group of real numbers and by \mathbb{Z} the subgroup consisting of the integers. The group \mathbb{T} is defined as the quotient $\mathbb{R}/2\pi\mathbb{Z}$ where, as indicated by the notation, $2\pi\mathbb{Z}$ is the group of the integral multiples of 2π . There is an obvious identification between functions on \mathbb{T} and 2π -periodic functions on \mathbb{R} , which allows an implicit introduction of notions such as continuity, differentiability, etc., for functions on \mathbb{T} . The *Lebesgue measure* on \mathbb{T} can be equally defined by means of the preceding identification: a function f is integrable on \mathbb{T} if the corresponding 2π -periodic function, which we denote again by f , is integrable on $[0, 2\pi)$ and we set

$$\int_{\mathbb{T}} f(t)dt = \int_0^{2\pi} f(x)dx.$$

In other words, we consider the interval $[0, 2\pi)$ as a model for \mathbb{T} and the Lebesgue measure dt on \mathbb{T} is the restriction of the Lebesgue measure of \mathbb{R} to $[0, 2\pi)$. The total mass of dt on \mathbb{T} is equal to 2π and many of our formulas would be simpler if we normalized dt to have total mass 1, that is, if we replace it by $dx/2\pi$. Taking intervals on \mathbb{R} as "models" for \mathbb{T} is very convenient, however, and we choose to put $dt = dx$ in order to avoid confusion. We "pay" by having to write the factor $1/2\pi$ in front of every integral.

An all-important property of dt on \mathbb{T} is its *translation invariance*, that is, for all $t_0 \in \mathbb{T}$ and f defined on \mathbb{T} ,

$$\int f(t - t_0)dt = \int f(t)dt^\dagger.$$

[†]Throughout this chapter, integrals with unspecified limits of integration are taken over \mathbb{T} .

1 FOURIER COEFFICIENTS

1.1 We denote by $L^1(\mathbb{T})$ the space of all (equivalence[†] classes of) complex-valued, Lebesgue integrable functions on \mathbb{T} . For $f \in L^1(\mathbb{T})$ we put

$$\|f\|_{L^1} = \frac{1}{2\pi} \int_{\mathbb{T}} |f(t)| dt.$$

It is well known that $L^1(\mathbb{T})$, with the norm so defined, is a Banach space.

DEFINITION: A *trigonometric polynomial* on \mathbb{T} is an expression of the form

$$(1.1) \quad P \sim \sum_{n=-N}^N a_n e^{int}.$$

The numbers n appearing in (1.1) are called the frequencies of P ; the largest integer n such that $|a_n| + |a_{-n}| \neq 0$ is called *the degree of P* . The values assumed by the index n are integers so that each of the summands in (1.1) is a function on \mathbb{T} . Since (1.1) is a finite sum, it represents a function, which we denote again by P , defined for each $t \in \mathbb{T}$ by

$$(1.2) \quad P(t) = \sum_{n=-N}^N a_n e^{int}.$$

Let P be defined by (1.2). Knowing the function P we can compute the coefficients a_n by the formula

$$(1.3) \quad a_n = \frac{1}{2\pi} \int P(t) e^{-int} dt$$

which follows immediately from the fact that for integers j ,

$$\frac{1}{2\pi} \int e^{ijt} dt = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{if } j \neq 0. \end{cases}$$

Thus we see that the function P determines the expression (1.1) and there seems to be no point in keeping the distinction between the expression (1.1) and the function P ; we shall consider trigonometric polynomials as both formal expressions and functions.

[†] $f \sim g$ if $f(t) = g(t)$ almost everywhere.

1.2 DEFINITION: A *trigonometric series* on \mathbb{T} is an expression of the form

$$(1.4) \quad S \sim \sum_{n=-\infty}^{\infty} a_n e^{int}.$$

Again, n assumes integral values; however, the number of terms in (1.4) may be infinite and there is no assumption whatsoever about the size of the coefficients or about convergence. The conjugate[†] of the series (1.4) is, by definition, the series

$$\tilde{S} \sim \sum_{n=-\infty}^{\infty} -i \operatorname{sgn}(n) a_n e^{int},$$

where $\operatorname{sgn}(n) = 0$ if $n = 0$ and $\operatorname{sgn}(n) = n/|n|$ otherwise.

1.3 Let $f \in L^1(\mathbb{T})$. Motivated by (1.3) we define the n th Fourier coefficient of f by

$$(1.5) \quad \hat{f}(n) = \frac{1}{2\pi} \int f(t) e^{-int} dt.$$

DEFINITION: The *Fourier series* $S[f]$ of a function $f \in L^1(\mathbb{T})$ is the trigonometric series

$$S[f] \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{int}.$$

The series conjugate to $S[f]$ will be denoted by $\tilde{S}[f]$ and referred to as the *conjugate Fourier series* of f . We shall say that a trigonometric series is a Fourier series if it is the Fourier series of some $f \in L^1(\mathbb{T})$.

1.4 We turn to some elementary properties of Fourier coefficients.

Theorem. Let $f, g \in L^1(\mathbb{T})$, then

$$(a) \quad \widehat{(f+g)}(n) = \hat{f}(n) + \hat{g}(n).$$

(b) For any complex number α

$$\widehat{(\alpha f)}(n) = \alpha \hat{f}(n).$$

[†]See Chapter III for motivation of the terminology.

(c) If \bar{f} is the complex conjugate[‡] of f then $\hat{\bar{f}}(n) = \overline{\hat{f}(-n)}$.

(d) Denote $f_\tau(t) = f(t - \tau)$, $\tau \in \mathbb{T}$; then

$$\hat{f}_\tau(n) = \hat{f}(n)e^{-in\tau}.$$

(e) $|\hat{f}(n)| \leq \frac{1}{2\pi} \int |f(t)| dt = \|f\|_{L^1}$.

The proofs of (a) through (e) follow immediately from (1.5) and the details are left to the reader.

1.5 Corollary. If $f_j \in L^1(\mathbb{T})$, $j = 0, 1, \dots$, and $\|f_j - f_0\|_{L^1} \rightarrow 0$, then $\hat{f}_j(n) \rightarrow \hat{f}_0(n)$ uniformly.

1.6 Theorem. Let $f \in L^1(\mathbb{T})$, assume $\hat{f}(0) = 0$, and define

$$F(t) = \int_0^t f(\tau) d\tau.$$

Then F is continuous, 2π -periodic, and

$$(1.6) \quad \hat{F}(n) = \frac{1}{in} \hat{f}(n), \quad n \neq 0.$$

PROOF: The continuity (and, in fact, the absolute continuity) of F is evident. The periodicity follows from

$$F(t + 2\pi) - F(t) = \int_t^{t+2\pi} f(\tau) d\tau = 2\pi \hat{f}(0) = 0,$$

and (1.6) is obtained through integration by parts:

$$\hat{F}(n) = \frac{1}{2\pi} \int_0^{2\pi} F(t) e^{-int} dt = \frac{-1}{2\pi} \int_0^{2\pi} F'(t) \frac{1}{-in} e^{-int} dt = \frac{1}{in} \hat{f}(n). \quad \blacktriangleleft$$

1.7 We now define the convolution operation in $L^1(\mathbb{T})$. The reader will notice the use of the group structure of \mathbb{T} and of the invariance of dt in the subsequent proofs.

[‡]Defined by: $\bar{f}(t) = \overline{f(t)}$ for all $t \in \mathbb{T}$.

Theorem. Let $f, g \in L^1(\mathbb{T})$. For almost all t , the function $f(t - \tau)g(\tau)$ is integrable (as a function of τ on \mathbb{T}) and, if we write

$$(1.7) \quad h(t) = \frac{1}{2\pi} \int f(t - \tau)g(\tau)d\tau,$$

then $h \in L^1(\mathbb{T})$ and

$$(1.8) \quad \|h\|_{L^1} \leq \|f\|_{L^1} \|g\|_{L^1}.$$

Moreover

$$(1.9) \quad \hat{h}(n) = \hat{f}(n)\hat{g}(n) \quad \text{for all } n.$$

PROOF: The functions $f(t - \tau)$ and $g(\tau)$, considered as functions of the two variables (t, τ) , are clearly measurable, hence so is

$$F(t, \tau) = f(t - \tau)g(\tau).$$

For every τ , $F(t, \tau)$ is just a constant multiple of f_τ , hence integrable dt , and

$$\frac{1}{2\pi} \int \left(\frac{1}{2\pi} \int |F(t, \tau)| dt \right) d\tau = \frac{1}{2\pi} \int |g(\tau)| \|f\|_{L^1} d\tau = \|f\|_{L^1} \|g\|_{L^1}.$$

Hence, by the theorem of Fubini, $f(t - \tau)g(\tau)$ is integrable (over $(0, 2\pi)$) as a function of τ for almost all t , and

$$\begin{aligned} \frac{1}{2\pi} \int |h(t)| dt &= \frac{1}{2\pi} \int \left| \frac{1}{2\pi} \int F(t, \tau) d\tau \right| dt \leq \frac{1}{4\pi^2} \iint |F(t, \tau)| dt d\tau \\ &= \|f\|_{L^1} \|g\|_{L^1}, \end{aligned}$$

which establishes (1.8). In order to prove (1.9) we write

$$\begin{aligned} \hat{h}(n) &= \frac{1}{2\pi} \int h(t)e^{-int} dt = \frac{1}{4\pi^2} \iint f(t - \tau)e^{-in(t-\tau)}g(\tau)e^{-in\tau} dt d\tau \\ &= \frac{1}{2\pi} \int f(t)e^{-int} dt \cdot \frac{1}{2\pi} \int g(\tau)e^{-in\tau} d\tau = \hat{f}(n)\hat{g}(n). \end{aligned}$$

As above, the change in the order of integration is justified by Fubini's theorem. ◀

1.8 DEFINITION: The *convolution* $f * g$ of the $(L^1(\mathbb{T}))$ functions f and g is the function h defined by (1.7). Using the star notation for the convolution, we can write (1.9) as

$$(1.10) \quad \widehat{f * g}(n) = \hat{f}(n)\hat{g}(n).$$

Theorem. *The convolution operation in $L^1(\mathbb{T})$ is commutative, associative, and distributive (with respect to the addition).*

PROOF: The change of variable $\vartheta = t - \tau$ gives

$$\frac{1}{2\pi} \int f(t - \tau)g(\tau)d\tau = \frac{1}{2\pi} \int g(t - \vartheta)f(\vartheta)d\vartheta,$$

that is,

$$f * g = g * f.$$

If $f_1, f_2, f_3 \in L^1(\mathbb{T})$, then

$$\begin{aligned} [(f_1 * f_2) * f_3](t) &= \frac{1}{4\pi^2} \iint f_1(t - u - \tau)f_2(u)f_3(\tau)du d\tau \\ &= \frac{1}{4\pi^2} \iint f_1(t - \omega)f_2(\omega - \tau)f_3(\tau)d\omega d\tau = [f_1 * (f_2 * f_3)](t). \end{aligned}$$

Finally, the distributive law

$$f_1 * (f_2 + f_3) = f_1 * f_2 + f_1 * f_3$$

is evident from (1.7). ◀

1.9 Lemma. *Assume $f \in L^1(\mathbb{T})$ and let $\varphi(t) = e^{int}$ for some integer n . Then*

$$(\varphi * f)(t) = \hat{f}(n)e^{int}.$$

PROOF:

$$(\varphi * f)(t) = \frac{1}{2\pi} \int e^{in(t-\tau)}f(\tau)d\tau = e^{int} \frac{1}{2\pi} \int f(\tau)e^{-in\tau} d\tau. \quad \blacktriangleleft$$

Corollary. *If $f \in L^1(\mathbb{T})$ and $k(t) = \sum_{-N}^N a_n e^{int}$, then*

$$(1.11) \quad (k * f)(t) = \sum_{-N}^N a_n \hat{f}(n)e^{int}.$$

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EXERCISES FOR SECTION 1

1.1. Compute the Fourier coefficients of the following functions (defined by their values on $[-\pi, \pi)$):

$$(a) \quad f(t) = \begin{cases} \sqrt{2\pi} & |t| < \frac{1}{2} \\ 0 & \frac{1}{2} \leq |t| \leq \pi. \end{cases}$$

$$(b) \quad \Delta(t) = \begin{cases} 1 - |t| & |t| < 1 \\ 0 & 1 \leq |t| \leq \pi. \end{cases}$$

What relation do you see between f and Δ ?

$$(c) \quad g(t) = \begin{cases} 1 & -1 < t \leq 0 \\ -1 & 0 < t < 1 \\ 0 & 1 \leq |t|. \end{cases}$$

What relation do you see between g and Δ ?

$$(d) \quad h(t) = t \quad -\pi < t < \pi.$$

1.2. Remembering Euler's formulas

$$\cos t = \frac{1}{2}(e^{it} + e^{-it}), \quad \sin t = \frac{1}{2i}(e^{it} - e^{-it}),$$

or

$$e^{it} = \cos t + i \sin t,$$

show that the Fourier series of a function $f \in L^1(\mathbb{T})$ is formally equal to

$$\frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos nt + B_n \sin nt)$$

where $A_n = \hat{f}(n) + \hat{f}(-n)$ and $B_n = i(\hat{f}(n) - \hat{f}(-n))$. Equivalently:

$$A_n = \frac{1}{\pi} \int f(t) \cos nt \, dt$$

$$B_n = \frac{1}{\pi} \int f(t) \sin nt \, dt.$$

Show also that if f is real valued, then A_n and B_n are all real; if f is even, that is, if $f(t) = f(-t)$, then $B_n = 0$ for all n ; and if f is odd, that is, if $f(t) = -f(-t)$, then $A_n = 0$ for all n .

1.3. Show that if $S \sim \sum a_j \cos jt$, then $\tilde{S} \sim \sum a_j \sin jt$.

1.4. Let $f \in L^1(\mathbb{T})$ and let $P(t) = \sum_{-N}^N a_n e^{int}$. Compute the Fourier coefficients of the function fP .

1.5. Let $f \in L^1(\mathbb{T})$, let m be a positive integer, and write

$$f_{(m)}(t) = f(mt).$$

Show that

$$\widehat{f_{(m)}}(n) = \begin{cases} \widehat{f}(\frac{n}{m}) & \text{if } m \mid n, \\ 0 & \text{if } m \nmid n. \end{cases}$$

1.6. The trigonometric polynomial $\cos nt = \frac{1}{2}(e^{int} + e^{-int})$ is of degree n and has $2n$ zeros on \mathbb{T} . Show that no trigonometric polynomial of degree $n > 0$ can have more than $2n$ zeros on \mathbb{T} .

Hint: Identify $\sum_{-n}^n a_j e^{ijt}$ on \mathbb{T} with $z^{-n} \sum_{-n}^n a_j z^{n+j}$ on $|z| = 1$.

1.7. Denote by C^* the multiplicative group of complex numbers different from zero. Denote by T^* the subgroup of all $z \in C^*$ such that $|z| = 1$. Prove that if G is a subgroup of C^* which is compact (as a set of complex numbers), then $G \subseteq T^*$.

1.8. Let G be a compact proper subgroup of \mathbb{T} . Prove that G is finite and determine its structure.

Hint: Show that G is discrete.

1.9. Let G be an infinite subgroup of \mathbb{T} . Prove that G is dense in \mathbb{T} .

Hint: The closure of G in \mathbb{T} is a compact subgroup.

1.10. Let α be an irrational multiple of 2π . Prove that $\{n\alpha \pmod{2\pi}\}_{n \in \mathbb{Z}}$ is dense in \mathbb{T} .

1.11. Prove that a continuous homomorphism of \mathbb{T} into C^* is necessarily given by an exponential function.

Hint: Use Exercise 1.7 to show that the map is into T^* ; determine the mapping on "small" rational multiples of 2π and use Exercise 1.9.

1.12. If E is a subset of \mathbb{T} and $\tau_0 \in \mathbb{T}$, we define $E + \tau_0 = \{t + \tau_0 : t \in E\}$; and say that E is invariant under translation by τ if $E = E + \tau$. Show that, given a set E , the set of $\tau \in \mathbb{T}$ such that E is invariant under translation by τ is a subgroup of \mathbb{T} . Hence prove that if E is a measurable set on \mathbb{T} and E is invariant under translation by infinitely many $\tau \in \mathbb{T}$, then either E or its complement has measure zero.

Hint: A set E of positive measure has points of density, that is, points τ such that $(2\varepsilon)^{-1} |E \cap (\tau - \varepsilon, \tau + \varepsilon)| \rightarrow 1$ as $\varepsilon \rightarrow 0$. ($|E_0|$ denotes the Lebesgue measure of E_0 .)

1.13. If E and F are subsets of \mathbb{T} , we write

$$E + F = \{t + \tau : t \in E, \tau \in F\}$$

and call $E + F$ the *algebraic sum* of E and F . We define similarly the sum of any finite number of sets. A set E is called a *basis for \mathbb{T}* if there exists an integer N such that $E + E + \cdots + E$ (N times) is \mathbb{T} . Prove that every set E of positive measure on \mathbb{T} is a basis.

Hint: Prove that if E contains an interval it is a basis. Using points of density prove that if E has positive measure then $E + E$ contains intervals.

1.14. Show that measurable proper subgroups of \mathbb{T} have measure zero.

1.15. Show that measurable homomorphisms of \mathbb{T} into C^* map it into T^* .

1.16. Let f be a measurable homomorphism of \mathbb{T} into T^* . Show that for all values of n , except possibly one value, $\hat{f}(n) = 0$.

2 SUMMABILITY IN NORM AND HOMOGENEOUS BANACH SPACES ON \mathbb{T}

2.1 We have defined the Fourier series of a function $f \in L^1(\mathbb{T})$ as a certain (formal) trigonometric series. The reader may wonder what is the point in the introduction of such formal series. After all, there is no more information in the (formal) expression $\sum_{-\infty}^{\infty} \hat{f}(n)e^{int}$ than there is in the simpler one $\{\hat{f}(n)\}_{-\infty}^{\infty}$ or the even simpler \hat{f} (with the understanding that the function \hat{f} is defined on the integers). As we shall see, both expressions, $\sum \hat{f}(n)e^{int}$ and \hat{f} , have their advantages; the main advantages of the series notation is that it indicates the way in which f can be reconstructed from \hat{f} . Much of this chapter and all of Chapter II will be devoted to clarifying the sense in which $\sum \hat{f}(n)e^{int}$ represents f . In this section we establish some of the main facts: we shall see that \hat{f} determines f uniquely and we show how we can find f if we know \hat{f} .

Two very important properties of the Banach space $L^1(\mathbb{T})$ are the following:

H-1' If $f \in L^1(\mathbb{T})$ and $\tau \in \mathbb{T}$, then

$$f_{\tau}(t) = f(t - \tau) \in L^1(\mathbb{T}) \quad \text{and} \quad \|f_{\tau}\|_{L^1} = \|f\|_{L^1}.$$

H-2' The $L^1(\mathbb{T})$ -valued function $\tau \mapsto f_{\tau}$ is continuous on \mathbb{T} , that is, for $f \in L^1(\mathbb{T})$ and $\tau_0 \in \mathbb{T}$

$$(2.1) \quad \lim_{\tau \rightarrow \tau_0} \|f_{\tau} - f_{\tau_0}\|_{L^1} = 0.$$

We shall refer to (H-1') as the translation invariance of $L^1(\mathbb{T})$; it is an immediate consequence of the translation invariance of the measure

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dt. In order to establish (H-2') we notice first that (2.1) is clearly valid if f is a continuous function. Remembering that the continuous functions are dense in $L^1(\mathbb{T})$, we now consider an arbitrary $f \in L^1(\mathbb{T})$ and $\varepsilon > 0$. Let g be continuous on \mathbb{T} such that $\|g - f\|_{L^1} < \varepsilon/2$; then

$$\begin{aligned} \|f_\tau - f_{\tau_0}\|_{L^1} &\leq \|f_\tau - g_\tau\|_{L^1} + \|g_\tau - g_{\tau_0}\|_{L^1} + \|g_{\tau_0} - f_{\tau_0}\|_{L^1} \\ &= \|(f - g)_\tau\|_{L^1} + \|g_\tau - g_{\tau_0}\|_{L^1} + \|(g - f)_{\tau_0}\|_{L^1} \leq \varepsilon + \|g_\tau - g_{\tau_0}\|_{L^1}. \end{aligned}$$

Hence $\overline{\lim}_{\tau \rightarrow \tau_0} \|f_\tau - f_{\tau_0}\|_{L^1} < \varepsilon$ and, ε being an arbitrary positive number, (H-2') is established.

2.2 DEFINITION: A *summability kernel* is a sequence $\{k_n\}$ of continuous 2π -periodic functions satisfying:

$$(S-1) \quad \frac{1}{2\pi} \int k_n(t) dt = 1.$$

$$(S-2) \quad \frac{1}{2\pi} \int |k_n(t)| dt \leq \text{const.}$$

(S-3) For all $0 < \delta < \pi$,

$$\lim_{n \rightarrow \infty} \int_\delta^{2\pi - \delta} |k_n(t)| dt = 0.$$

A *positive summability kernel* is one such that $k_n(t) \geq 0$ for all t and n . For positive kernels the assumption (S-2) is clearly redundant.

We consider also families k_r depending on a continuous parameter r instead of the discrete n . Thus the Poisson kernel $\mathbf{P}(r, t)$ (which we shall define at the end of this section) is defined for $0 \leq r < 1$ and we replace in (S-3), as well as in the applications, the limit “ $\lim_{n \rightarrow \infty}$ ” by “ $\lim_{r \rightarrow 1}$ ”.

The following lemma is stated in terms of vector-valued integrals. We refer to Appendix A for the definition and relevant properties.

Lemma. Let B be a Banach space, φ a continuous B -valued function on \mathbb{T} , and $\{k_n\}$ a summability kernel. Then:

$$\lim_{n \rightarrow \infty} \frac{1}{2\pi} \int k_n(\tau) \varphi(\tau) d\tau = \varphi(0).$$