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Reductive groups as metric spaces

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1. Introduction

In this paper four descriptions of one and the same quasi-isometry class of pseudo-metrics on a reductive group G over a local field are given. They are as follows. The first one is the word metric corresponding to a compact set of generators of G. The second one is the pseudo-metric given by the action of G by isometries on a metric space. That these two pseudo-metrics on a group G are quasi-isometric holds in great generality. The third pseudo-metric is defined using the operator norm for a representation ρ of G. This pseudo-metric depends very much on the representation. But for a reductive group over a local field it does not up to quasi-isometry. The fourth pseudo-metric is given on a split torus over a local field K by valuations of the K^*–factors. The main result is that these four pseudo-metrics on a reductive group over a local field coincide up to quasi-isometry. We thus have four different descriptions of one and the same very natural and distinguished quasi-isometry class of pseudo-metrics.

This paper contains foundational material for joint work in progress with G. A. Margulis on the following two topics. One is work on the following question of C. L. Siegel's. Given a reductive group G over a local field and an S–arithmetic subgroup Γ of G, it is one of the main results of reduction theory to describe a fundamental domain R for Γ in G, a so called *Siegel domain*. Siegel asked in his Japan lectures [8, Sect. 10] on reduction theory of 1959, if – in our terminology, see Section 2.3 – the natural map $R \to \Gamma\backslash G$ is a coarse isometry. He asked this question only for the special case $G = SL(n, \mathbb{R})$, $\Gamma = SL(n, \mathbb{Z})$ and d the pseudo-metric on G coming from the standard Riemannian metric on the symmetric space of G, the space of positive definite real symmetric $n \times n$–matrices. We now have a positive answer in full generality, for arbitrary reductive groups G over local fields, S–arithmetic subgroups Γ and for pseudo-metrics d on G which are norm-like. We call a pseudo-metric on G *norm–like* if it is coarsely isometric to a metric coming from the operator norm of a rational representation, or, equivalently, coming from a norm on a maximal split torus, see Sections 5 and 6. This raises of course the question which pseudo-metrics are norm-like. Note that coarse isometry is a much stricter equivalence relation among pseudo-metrics than quasi-isometry. We show in this paper that the three last types of pseudo-metrics on reductive groups are norm-like. It is an open question whether the first one, namely the word metric, or, more generally (Section 3.8), any coarse path pseudo-metric, gives a norm-like pseudo-metric. In joint work in progress with G. A. Margulis we show that this is the case if G is a torus or if the rank r of a maximal split torus in the semi-simple part of G is equal to one. This is probably even true for $r = 2$.[1]

The question of Siegel has an interesting history. A first positive answer was given by Borel in [1]. It was discovered much later (see the notes to paper [1] in Borel's Oeuvres vol. IV) that the proof contains a gap. It occurs on pp. 550 – 552, (12) does not imply (14), but (14) is essential to prove (5), the main inequality. There are now proofs for Siegel's conjecture, in its original form [2] and more generally for real reductive groups G, ordinary arithmetic subgroups and the pseudo-metric d coming from the symmetric space [4, 6].

Here are some more details about our approach to Siegel's question. For the sake of exposition we restrict ourselves to the case $G = SL(n, \mathbb{R})$ and $\Gamma = SL(n, \mathbb{Z})$. Let T be the subgroup of $SL(n, \mathbb{R})$ of diagonal matrices $t = \mathrm{diag}(t_1, \ldots, t_n)$ of determinant one, a maximal $\mathbb{R}$–split torus. The *negative Weyl chamber* is by definition the subset $C^- = \{\mathrm{diag}(t_1, \ldots, t_n) \in T \mid 0 < t_1 \leq t_2 \leq \cdots \leq t_n\}$. A *Siegel set* R in $SL(n, \mathbb{R})$ is, by definition, a subset of G of the form $K \cdot C^- \cdot L$, where K and L are compact subsets of G. The main result of reduction theory for this case states that for appropriate sets K and L the Siegel set R is a set of representatives for G/Γ. So the natural map $G \to G/\Gamma$ restricts to a surjection $\pi : R \to G/\Gamma$. It has other nice properties, e.g., $\pi\mid_R$ is a proper map. The question of Siegel mentioned above asked about the metric properties of π. Let d be a right invariant pseudo-metric on G. Define a pseudo-metric $\overline{d}$ on G/Γ in the natural way, i.e., $\overline{d}(g\,\Gamma, h\,\Gamma) = \inf\{d(g\gamma, h) \mid \gamma \in \Gamma\}$. Now Siegel's question was: is $\pi : R \to G/\Gamma$ a coarse isometry? In other words: is there a constant C such that

$$\overline{d}(g\,\Gamma, h\,\Gamma) \leq d(g, h) \leq \overline{d}(g\,\Gamma, h\,\Gamma) + C$$

for every pair g, h of points of R? Siegel himself showed in [8, Sect. 10] that this is the case if we fix one variable, that is, for every $g \in G$ there is a constant

[1] Note added in proof: this is even true in general; see forthcoming joint work with G. A. Margulis.

$C = C(g)$ such that the right inequality holds for every $h \in R$. It suffices to show this for one point $g \in G$.

Here are the main steps of our proof that the answer is yes. We may assume that g and h are in the negative Weyl chamber C^- and that $d = d^{\rho}_{\text{op}}$ is the metric coming from a rational representation, see Section 5. We prove that, for $\gamma \in \Gamma$,

$$d(g\,\gamma, h) \underset{\text{(I)}}{\to\geq} d(a(g\,\gamma), h) \underset{\text{(II)}}{\to\geq} d(w^{-1}g, h) \underset{\text{(III)}}{\to\geq} d(g, h)$$

up to constants, where $G = K \cdot A \cdot N$, $g = k(g) \cdot a(g) \cdot n(g)$, is the Iwasawa decomposition and $\gamma \in B\,w\,B$ in the Bruhat decomposition with w an element of the Weyl group S_{n-1}. Note that (III) is a very special property of reflection groups. It does for example not hold for g, h in the fundamental domain of a finite rotation group and w in this group. An important step in the proof of (II) is

$$(\text{II}')\ a(g\,\gamma) = w^{-1} g\, w + r$$

where r is up to a compact error term the exponential of a positive linear combination of $\Sigma_{w^{-1}}$ where $\Sigma_{w^{-1}} = \{\alpha \in \Phi^+ \mid w^{-1}\alpha\, w \in \Phi^+\}$ and Φ^+ is the set of positive roots. That we found (II′) is due to discussions with Alex Eskin who showed us a geometric picture of this fact.

Let us point out the following features of this proof. It is different from both Ding's [2] which is by induction on n, and from Leuzinger's [6] which uses Tits buildings and facts about the geometry of symmetric and locally symmetric spaces, in particular their geometry at infinity. Our proof works in full generality, for arbitrary local fields and arbitrary S–arithmetic subgroups. Also, we admit arbitrary norm-like metrics, not only those coming from the symmetric space or the Bruhat–Tits building. Finally it gives further information concerning reduction theory, namely the inequalities stated above.

2. Metrics

We first need to recall some concepts concerning metric spaces.

2.1. Let X be a set. A function $d : X \times X \to \mathbb{R}$ is called a *pseudo-metric* (on X) if d is non-negative, zero on the diagonal, symmetric and fulfills the triangle inequality, i.e., if

$$\begin{aligned} d(x, y) &\geq 0 \text{ for every } x, y \text{ in } X \\ d(x, x) &= 0 \text{ for every } x \text{ in } X \\ d(x, y) &= d(y, x) \text{ for every } x, y \text{ in } X \\ d(x, y) + d(y, z) &\geq d(x, z) \text{ for every } x, y, z \text{ in } X. \end{aligned}$$

So a pseudo-metric on X is a *metric* on X if and only if $d(x, y) = 0$ implies

$x = y$. A pair (X, d) consisting of a set X and a (pseudo-) metric d on X is called a *(pseudo-) metric space*. In a pseudo-metric space (X, d) the ball of radius r with center x is denoted $B_d(x, r)$ or $B(x, r)$. So

$$B_d(x,r) = \{y \in X : d(x,y) \leq r\}.$$

2.2. Let (X, d) and (X', d') be pseudo-metric spaces. A map $f : X \to X'$ is called a *quasi-isometry* if there are real numbers $C_1 > 0$ and C_2 such that

$$C_1^{-1} \cdot d(x, y) - C_2 \leq d'(f(x), f(y)) \leq C_1 \cdot d(x, y) + C_2$$

and $X' = \bigcup_{x \in X} B_{d'}(f(x), C_2)$. Thus, for every point $x' \in X'$ there is a point $x \in X$ such that $d(x', f(x)) \leq C_2$. Define a map $g : X' \to X$ by choosing for every $x' \in X$ a point $x = g(x')$ with this property. Then $g : X' \to X$ is a quasi-isometry, actually with the same multiplicative constant C_1, and we have $d(x, gf(x)) \leq C_2$ and $d'(x', fg(x')) \leq C_2$ for every $x \in X$ and $x' \in X'$.

2.3. A map $f : X \to X'$ between pseudo-metric spaces (X, d) and (X', d') is called a *coarse isometry* if f is a quasi-isometry and the multiplicative constant C_1 can be chosen to equal 1. Equivalently, the function $(x, y) \mapsto d'(f(x), f(y)) - d(x, y)$ is bounded on $X \times X$ and every point of X' is at bounded distance from $f(X)$. Finally, $f : X \to X'$ is called an *isometry* if both these bounds are zero, i.e., if f is surjective and $d'(f(x), f(y)) = d(x, y)$ for every x, y in X. If f is a (coarse) isometry, then so is any map $g : X' \to X$ considered above. It follows that if there is a (quasi-, coarse) isometry from X to X' then there is one from X' to X. Two pseudo-metrics on the same set are called *(quasi-, coarsely) isometric* if the identity map is a (quasi-, coarse) isometry. It follows that these relations are equivalence relations between pseudo-metric spaces and also between pseudo-metrics on the same set.

2.4. We will mainly be interested in pseudo-metrics on groups. So let G be a group. A pseudo-metric d on G will be called *left invariant (right invariant)* if every left translation (right translation) is an isometry. So d is left invariant on G if and only if $d(gh_1, gh_2) = d(h_1, h_2)$ for every g, h_1, h_2 in G. Define a function f on G by $f(g) = d(e, g)$. If d is a left (right) invariant pseudo-metric on G, then f is non-negative, zero at the identity element, symmetric and fulfills the triangle inequality, i.e.,

$$\begin{aligned} f(g) &\geq 0 && \text{for every } g \in G, \\ f(e) &= 0 && \text{for the identity element } e, \\ f(g) &= f(g^{-1}) && \text{for every } g \in G \text{ and} \\ f(gh) &\leq f(g) + f(h) && \text{for every } g, h \text{ in } G. \end{aligned}$$

Conversely, given a function f with these properties then $d(g, h) := f(g^{-1}h)$, resp. $d(g,h) = f(hg^{-1})$, defines the unique left (right) invariant pseudo-metric d on G such that $d(e,g) = f(g)$ for every $g \in G$. A function f on G with these properties is sometimes called a norm on G. But we want to reserve the term "norm" for a more special situation.

3. The word metric

Let G be a group and let Σ be a set of generators of G. Then the *word length* $\ell_\Sigma(g)$ of an element $g \in G$ with respect to Σ is defined as

$$\ell_\Sigma(g) = \inf\left\{r : g = a_1^{\varepsilon_1} \dots a_r^{\varepsilon_r},\ a_i \in \Sigma,\ \varepsilon_i \in \{+1,-1\}\right\}.$$

The function ℓ_Σ has the properties stated above and furthermore $\ell_\Sigma(g) = 0$ implies $g = e$. So $d_\Sigma(g,h) := \ell_\Sigma(g^{-1}h)$ defines a left invariant metric d_Σ on G, which is called the *word metric* associated with Σ. The ball of radius r with center e is

$$B_{d_\Sigma}(e,r) = (\Sigma \cup \Sigma^{-1})^r = \left\{a_1^{\varepsilon_1} \dots a_r^{\varepsilon_r} : a_i \in \Sigma\ ,\ \varepsilon_i \in \{+1,-1\}\right\},$$

and thus consists of all words of length at most r with respect to the alphabet $\Sigma \cup \Sigma^{-1}$. The word metric d_Σ depends of course on Σ. But if Σ and Σ' are both *finite* sets of generators of G then d_Σ and $d_{\Sigma'}$ are quasi-isometric, since if $\ell_\Sigma(\Sigma')$ is bounded by C_1 then $d_\Sigma \le C_1 \cdot d_{\Sigma'}$. Similarly:

3.1. Lemma. *Let G be a locally compact topological group and let Σ and Σ' be compact sets of generators of G. Then the word metrics d_Σ and $d_{\Sigma'}$ on G are quasi-isometric. They are actually Lipschitz equivalent, that is, the additive constant C_2 in the definition of quasi-isometry may be chosen equal to zero.*

By the preceding argument it suffices to show the following.

3.2. Lemma. *Let G be a locally compact topological group and let Σ be a compact set of generators of G. Then every compact subset of G has bounded word length ℓ_Σ.*

Proof. The sequence of compact subsets $A_n = B_{d_\Sigma}(e,n) = (\Sigma \cup \Sigma^{-1})^n$ of G covers the locally compact space G. So one of them contains a non-empty open subset U of G by the Baire category theorem, say $U \subset A_n$. Then A_{2n} is a neighbourhood of the identity element e, since A_{2n} contains $U \cdot U^{-1}$. If now K is a compact subset of G there is a finite subset M of K such that $M \cdot A_{2n}$ contains K. Thus $\ell_\Sigma(K) \le \ell_\Sigma(M) + 2n$. □

3.3. Remark. Both Lemmas 3.1 and 3.2 remain true if Σ and Σ' are relatively compact sets of generators of G which contain a non-empty open subset of G, as follows from the second part of the proof of Lemma 3.2. But Lemma 3.2, and hence Lemma 3.1, is not true for an arbitrary relatively compact set of generators of G; e.g., let G be the additive group $\mathbb{R}$. The word length $\ell_{\Sigma'}$ corresponding to the set of generators $\Sigma' = [0, 1]$ is $\ell_{\Sigma'}(x) = \lceil |x| \rceil$, the smallest integer $\geq |x|$. Consider the following set of generators Σ. There is a basis B of the $\mathbb{Q}$–vector space $\mathbb{R}$ such that $B \subset [0, 1]$ and B contains for every $n \in \mathbb{N}$ an element b_n with $0 \leq b_n \leq \frac{1}{n}$. Such a basis can be obtained from a given basis of $\mathbb{R}$ over $\mathbb{Q}$ by multiplying every basis element with an appropriate rational number. Put $\Sigma = \{q \cdot b : \ b \in B, \ q \in \mathbb{Q} \cap [0, 1]\} \subset [0, 1]$. Then Σ is a set of generators of $\mathbb{R}$, contained in $[0, 1]$ but ℓ_Σ is unbounded on $[0, 1]$, since $\ell_\Sigma(n\, b_n) = n$. In fact, for every real number $x = \sum_{b\in B} q_b \cdot b$ with $q_b \in \mathbb{Q}$, we have $\ell_\Sigma(x) = \sum_{b\in B} \lceil |q_b| \rceil$.

Here is a geometric approach to the word metric.

3.4. Definition. A pseudo-metric d on a set X is called a *coarse path pseudo-metric* if there is a real number C such that for every pair of points x, y in X there is a sequence $x = x_0, x_1, \ldots, x_t = y$ for which $d(x_{i-1}, x_i) \leq C$ for $i = 1, \ldots, t$ and

$$d(x, y) \geq \sum_{i=1}^{t} d(x_{i-1}, x_i) - C.$$

In other words, the triangle inequality $d(x, y) \leq \sum_{i=1}^{t} d(x_{i-1}, x_i)$ is in fact an equality up to a bounded error.

3.5. A left invariant pseudo-metric d on a group G is a coarse path pseudo-metric if and only if the function f with $f(g) = d(e, g)$ has the following property. There is a real number C such that for every $g \in G$ there is a sequence $g_1, \ldots, g_t$ of elements of G such that $g = g_1 \cdot \cdots \cdot g_t$, $f(g_i) \leq C$ for $i = 1, \ldots, t$ and $f(g) \geq \sum_{i=1}^{t} f(g_i) - C$. The equivalence is seen as follows. Starting with $g \in G$ take a sequence $x_0 = e, x_1, \ldots, x_t = g$ as above and put $g_i = x_{i-1}^{-1} \cdot x_i$. Conversely, for x, y in G take a sequence $g_1, \ldots, g_t$ as above for $g = x^{-1} y$ and put $x_i = x \cdot g_1 \cdot \cdots \cdot g_i$.

3.6. Example. A word metric d_Σ on a group is a coarse path metric, since by definition $C = 1$, $B(e, 1) = \Sigma \cup \Sigma^{-1} \cup \{e\}$ and the error in the triangle inequality is zero with notation as in 3.4.

3.7. One can generalize this example as follows. Given a set of generators Σ of G and a bounded function $\omega : \Sigma \to [0, \infty)$ on Σ we can define a *weighted* *word length* on G by

$$\ell_{\Sigma,\omega}(g) = \inf\left\{ \sum_{i=1}^{t} \omega(g_i) : \ t \in \mathbb{N} \cup \{0\},\ g = g_1^{\varepsilon_1} \dots g_t^{\varepsilon_t}, \right.$$
$$\left. g_i \in \Sigma,\ \varepsilon_i \in \{+1, -1\} \right\}.$$

Then $\ell_{\Sigma,\omega}$ has all the properties of 2.4 so that $d_{\Sigma,\omega}(g,h) := \ell_{\Sigma,\omega}(g^{-1}h)$ defines a left invariant pseudo-metric on G which is in fact a coarse path pseudo-metric, as is readily seen. Furthermore, $d_{\Sigma,\omega}$ is the supremum of the pseudo-metrics d on X with the property that $d(e,g) \leq \omega(g)$ for $g \in \Sigma$.

3.8. The importance of this generalization lies in the following fact: every left invariant coarse path pseudo-metric is a weighted word pseudo-metric up to coarse isometry. More precisely, let d be a left invariant coarse path pseudo-metric on G and let C be as in 3.4. Then $\Sigma := B_d(e, C)$ is a set of generators of G and we have

$$d_{\Sigma,\omega} - C \leq d \leq d_{\Sigma,\omega},$$

where $\omega : \Sigma \to [0, C]$ is defined by $\omega(g) = d(e,g)$ for $g \in \Sigma$.

3.9. Note that a metric d' that is coarsely isometric to a coarse path metric d need not be a coarse path metric itself; e.g., on $G = \mathbb{Z}$ the metric

$$d'(x,y) = \begin{cases} 0 & \text{if } x = y \\ |x-y|+1 & \text{if } x \neq y \end{cases}$$

is left invariant and coarsely isometric to the Euclidean metric on $\mathbb{Z}$ which is a left invariant coarse path metric, in fact the word metric for the set of generators $\Sigma = \{1\}$. But d' is not a coarse path metric. For a given $C > 0$ the error term $d_{\Sigma,\omega}(e,n) - d'(e,n)$ grows linearly in $|n|$, where $\Sigma = B(0, C)$ and $\omega(m) = d'(0,m)$. If we consider coarse path pseudo-metrics up to quasi-isometry only, we do not need a weight function ω by the following lemma.

3.10. Lemma. *Let d be a left invariant coarse path pseudo-metric on a group G. Then d is quasi-isometric to a word metric, namely to d_Σ with $\Sigma = B_d(e, C)$, where C is as in* 3.4.

Proof. Let C be a constant as in Definition 3.4, put $f(g) = d(e,g)$ and $\Sigma = B(e,C)$. Then for every $g \in G$ there are $g_1 \dots g_t \in \Sigma$ such that $g = g_1 \dots g_t$ and

$$f(g) \geq \sum_{i=1}^{t} f(g_i) - C. \qquad (*)$$

We may assume that for every $i = 1, \ldots, t-1$ we have $f(g_i) + f(g_{i+1}) > C$. Since if this is not the case we combine successive factors $g_i, g_{i+1}, \ldots, g_j$ into one factor $g_i \cdot \cdots \cdot g_j$ such that $f(g_i \cdot \cdots \cdot g_j) \leq C$ but $f(g_i \cdot \cdots \cdot g_j \cdot g_{j+1}) > C$, starting with $i = 1$. This does not destroy the validity of $(*)$ by the triangle inequality. Then

$$f(g) \geq \frac{t-1}{2} \cdot C - C.$$

This together with the obvious inequality $t \geq \ell_\Sigma(g)$ shows one of the inequalities of the desired quasi-isometry. The other one is seen as follows. The group G is generated by $\Sigma = B(e, C)$. Let $s = \ell_\Sigma(g)$, $g = g_1 \cdot \cdots \cdot g_s$, $g_i \in \Sigma$. Then $f(g) \leq \sum_{i=1}^{s} f(g_i) \leq C \cdot \ell_\Sigma(g)$. □

As a corollary we obtain the following uniqueness result.

3.11. Proposition. *Let G be a locally compact topological group. Let d and d′ be two left invariant coarse path pseudo-metrics on G with the following two properties:*

(C) *Compact sets have bounded diameter.*
(P) *Balls of bounded radius are relatively compact.*

Then d and d′ are quasi-isometric. Furthermore, if such a pseudo-metric d exists on G, then G has a compact set Σ of generators and d is quasi-isometric to the corresponding word metric d_Σ.

The letter C alludes to “compact” or “continuous”. Note that C holds if d is continuous, but that the pseudo-metrics we consider need not be continuous, e.g., d_Σ is not, in general. The letter P alludes to “proper” since (P) holds if and only if inverse images of compact sets have compact closure for the map $x \mapsto d(e, x)$ (or for the map $x \mapsto d(y, x)$ for some (any) point $y \in G$).

Proof. d is quasi-isometric to the word metric d_Σ with $\Sigma = B_d(e, C)$, where C is as in Definition 3.4 for d. Then Σ is relatively compact by (P) for d. Similarly for d' which is quasi-isometric to $d_{\Sigma'}$, with Σ' relatively compact. Then Σ' is of bounded diameter for d, by (C), hence for the quasi-isometric word metric d_Σ, too. Thus $\ell_\Sigma(\Sigma') \leq C_1$, say, which implies $d_\Sigma \leq C_1 \cdot d_{\Sigma'}$ and similarly for the converse. Finally, Σ is a relatively compact set of generators for G, so G has a compact set Σ'' of generators. It follows as above that $d_\Sigma \leq C_2 \cdot d_{\Sigma''}$ for some $C_2 > 0$ and the converse $d_{\Sigma''} \leq C_3 d_\Sigma$ for some $C_3 > 0$ by the Baire category argument of the proof of 3.2. □

4. A geometric pseudo-metric

4.1. Let (X, d) be a pseudo-metric space and let the group G act on X by isometries. Let x_0 be a point of X. Then

$$d_{X,x_0}(g, h) := d(g\,x_0, h\,x_0) \tag{4.1}$$

defines a left invariant pseudo-metric d_{X,x_0} on G. For another point $x_1 \in X$ the pseudo-metrics d_{X,x_0} and d_{X,x_1} on G are coarsely isometric, see 2.3. There are many examples of this type. Here are two of them. Another one is 4.5.

4.2. Let G be a connected Lie group. There is a left invariant Riemannian tensor on G which gives rise to a left invariant path metric d_{Riem} on G. Any two such metrics d_{Riem} are quasi-isometric, in fact Lipschitz equivalent, since any two norms on the finite dimensional real vector space T_eG are equivalent. This metric on G can be regarded as an example of a geometric pseudo-metric as above, if we let G act on (G, d_{Riem}) by left translations.

4.3. Let G be a Lie group, which is connected or, more generally, has a finite group of connected components, and let K be a maximal compact subgroup of G. Then the homogeneous space $X = G/K$ carries a Riemannian tensor invariant against the action of G on X. Thus, for the corresponding path metric d_X on X the group G acts by isometries. Again, d_X is unique up to Lipschitz equivalence. Hence the corresponding pseudo-metrics d_{X,x_0} on G are unique up to Lipschitz equivalence if we fix x_0, e.g. $x_0 = e \cdot K$, and are unique up to quasi-isometry for arbitrary $x_0 \in X$.

We ask if the pseudo-metrics 4.2 and 4.3 are quasi-isometric to each other and quasi-isometric to the word metric for a compact set of generators. The answer is yes by the following proposition in view of Lemma 3.1.

4.4. Proposition. *Let (X, d) be a locally compact space X with a coarse path pseudo-metric d having the properties (C) and (P) of* 3.11. *Suppose the locally compact group G acts properly on X by isometries such that $G\backslash X$ is compact. Then G has a compact set Σ of generators and the pseudo-metric d_{X,x_0} on G is quasi-isometric to the word metric d_Σ. It follows that any two pseudo-metrics of the form d_{X,x_0} for spaces X as above are quasi-isometric. (Recall that an action of a locally compact group G on a locally compact space X is called proper if for every compact subset K of X the subset $\{g \in G;\ g\,K \cap K \neq \emptyset\}$ of G is compact.)*

Proof. Let K be a compact subset of X such that $X = GK$. We may assume that $x_0 \in K$. Let D be the diameter of K. Then every point of X is of distance $\leq D$ from some point of the orbit $G\,x_0$. Thus the embedding of the orbit $G\,x_0$

into X is a coarse isometry from $(G\,x_0, d\,|_{G\,x_0})$ to (X, d). It follows that the G–map $G \to X$, $g \mapsto g\,x_0$, is a coarse isometry since it is the composition of the isometry $G \to G\,x_0$, $g \mapsto g\,x_0$, and the coarse isometry $G\,x_0 \to X$. Note that the pseudo-metric d_{X,x_0} on G also has the properties (C) and (P). To see (P) use that the action of G on X is proper. Let C be a constant as in the Definition 3.4 of a coarse path pseudo-metric. Let $\Sigma = B(e, C + 2D)$ with respect to the pseudo-metric d_{X,x_0} on G. Then Σ is relatively compact. We may assume that Σ is a neighbourhood of e by taking a larger constant C or D if necessary. We claim that Σ generates G and that d_{X,x_0} and d_Σ are quasi-isometric which implies our claim in view of Remark 3.3. To prove this let $g \in G$. There is a coarse path $x_0, x_1, \ldots, g\,x_0 = x_t$ such that $d(x_{i-1}, x_i) \leq C$ for $i = 1, \ldots, t$ and

$$d(x_0, g\,x_0) \geq \sum_{i=1}^{t} d\,(x_{i-1}, x_i) - C.$$

For every $i = 0, \ldots, t$ there is an element $g_i \in G$ such that $d(x_i, g_i\,x_0) \leq D$. Here we put $g_0 = e$ and $g_t = g$. Then $d_{X,x_0}(g_{i-1}, g_i) = d(g_{i-1}x_0, g_i x_0) \leq C + 2D$, so $g_{i-1}^{-1} \cdot g_i \in \Sigma$ and hence g is in the group generated by Σ. Furthermore, we may assume that in our coarse path we have $d(x_{i-1}, x_i) + d(x_i, x_{i+1}) > C$ for $i = 1, \ldots, t-1$, since otherwise we leave out some points of our coarse path. It follows that

$$d_{X,x_0}(e, g) \geq \sum_{i=1}^{t} d(x_{i-1}, x_i) - C \geq \frac{t-1}{2} \cdot C - C$$

and thus

$$d_\Sigma(e, g) \leq t \leq \frac{2}{C} d_{X,x_0}(e, g) + 3.$$

The inverse inequality is easy to see, as follows. Let $g = g_1 \ldots g_t$, $g_i \in \Sigma$ with $t = \ell_\Sigma(g)$. Note that $\Sigma = \Sigma^{-1}$, so we can avoid factors of the form g_i^{-1}. Then

$$d_{X,x_0}(e, g) = d(x_0, g\,x_0) \leq \sum_{i=1}^{t} d(h_{i-1}x_0, h_i x_0)$$

where $h_i = g_1 \ldots g_i$. Thus $h_t = g$, $h_0 = e$ and $h_{i-1}^{-1} h_i = g_i \in \Sigma$ and hence

$$d(h_{i-1}x_0, h_i\,x_0) = d(x_0, h_{i-1}^{-1} h_i\,x_0) = d(x_0, g_i\,x_0) \leq C + 2D$$

which implies $d_{X,x_0}(e, g) \leq (C + 2D)t = (C + 2D)\ell_\Sigma(g) = (C + 2D)d_\Sigma(e, g)$. □

4.5. Example. For a non-archimedean local field K the group $G = \underline{G}(K)$ of K–points of a simple algebraic group $\underline{G}$ defined over K acts on the corresponding