Introduction

SIR MICHAEL ATIYAH

It is a great pleasure for me to write this introduction to the volume celebrating Graeme Segal's 60th birthday. Graeme was one of my first Ph.D. students but he rapidly moved on to become a collaborator and colleague. Over the years we have written a number of joint papers, but the publications are merely the tide-marks of innumerable discussions. My own work has been subtly influenced by Graeme's point of view: teacher and student can and do interchange roles, each educating the other.

Graeme has a very distinctive style. For him brevity is indeed the soul of wit, arguments should be elegant and transparent, lengthy calculations are a sign of failure and algebra should be kept firmly in its place. He only publishes when he is ready, when he is satisfied with the final product. At times this perfectionist approach means that his ideas, which he generously publicizes, get absorbed and regurgitated by others in incomplete form. But his influence is widely recognized, even when the actual publication is long-delayed.

Topology has always been at the heart of Graeme's interests, but he has interpreted this broadly and found fruitful pastures as far away as theoretical physics. There was a time when such deviation from the strict path of pure topology was deemed a misdemeanour, particularly when the field into which Graeme deviated was seen as less than totally rigorous. But time moves on and subsequent developments have fully justified Graeme's 'deviance'. He is one of a small number of mathematicians who have had an impact on theoretical physicists.

Of all his works I would single out his beautiful book on Loop Groups, written jointly with his former student Andrew Pressley. In a difficult area, straddling algebra, geometry, analysis and physics the book manages to maintain a coherent outlook throughout, and it does so with style. It is a real

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treasure, a worthy successor in its way to Hermann Weyl's *The Classical Groups*.

Perhaps we should look forward to another book in the same mould – in time for the 70th birthday?

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Part I

Contributions

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1

A variant of K-theory: $K_{\pm}$

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1 Introduction

Topological K-theory [2] has many variants which have been developed and exploited for geometric purposes. There are real or quaternionic versions, 'Real' K-theory in the sense of [1], equivariant K-theory [14] and combinations of all these.

In recent years K-theory has found unexpected application in the physics of string theories [6] [12] [13] [16] and all variants of K-theory that had previously been developed appear to be needed. There are even variants, needed for the physics, which had previously escaped attention, and it is one such variant that is the subject of this paper.

This variant, denoted by $K_{\pm}(X)$, was introduced by Witten [16] in relation to 'orientifolds'. The geometric situation concerns a manifold X with an involution τ having a fixed sub-manifold Y. On X one wants to study a pair of complex vector bundles (E^+, E^-) with the property that τ interchanges them. If we think of the virtual vector bundle $E^+ - E^-$, then τ takes this into its negative, and $K_{\pm}(X)$ is meant to be the appropriate K-theory of this situation.

In physics, X is a 10-dimensional Lorentzian manifold and maps $\Sigma \to X$ of a surface Σ describe the world-sheet of strings. The symmetry requirements for the appropriate Feynman integral impose conditions that the putative K-theory $K_{\pm}(X)$ has to satisfy.

The second author proposed a precise topological definition of $K_{\pm}(X)$ which appears to meet the physics requirements, but it was not entirely clear how to link the physics with the geometry.

In this paper we elaborate on this definition and also a second (but equivalent) definition of $K_{\pm}(X)$. Hopefully this will bring the geometry and physics closer together, and in particular link it up with the analysis of Dirac operators.

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Although $K_{\pm}(X)$ is defined in the context of spaces with involution it is rather different from Real K-theory or equivariant K-theory (for $G = Z_2$), although it has superficial resemblances to both. The differences will become clear as we proceed, but at this stage it may be helpful to consider the analogy with cohomology. Equivariant cohomology can be defined (for any compact Lie group G), and this has relations with equivariant K-theory. But there is also 'cohomology with local coefficients', where the fundamental group $\pi_1(X)$ acts on the abelian coefficient group. In particular for integer coefficients Z the only such action is via a homomorphism $\pi_1(X) \to Z_2$, i.e. by an element of $H^1(X; Z_2)$ or equivalently a double-covering $\tilde{X}$ of X.

This is familiar for an unoriented manifold with $\tilde{X}$ its oriented double-cover. In this situation, if say X is a compact n-dimensional manifold, then we do not have a fundamental class in $H^n(X; Z)$ but in $H^n(X; \tilde{Z})$ where $\tilde{Z}$ is the local coefficient system defined by $\tilde{X}$. This is also called 'twisted cohomology'.

Here $\tilde{X}$ has a fixed-point-free involution τ and, in such a situation, our group $K_{\pm}(\tilde{X})$ is the precise K-theory analogue of twisted cohomology. This will become clear later.

In fact K-theory has more sophisticated twisted versions. In [8] Donovan and Karoubi use Wall's graded Brauer group [15] to construct twistings from elements of $H^1(X; Z_2) \times H^3(X; Z)_{\text{torsion}}$. More general twistings of K-theory arise from automorphisms of its classifying space, as do twistings of equivariant K-theory. Among these are twistings involving a general element of $H^3(X; Z)$ (i.e., one which is not necessarily of finite order). These are also of interest in physics, and have recently been the subject of much attention [3] [5] [9]. Our $K_{\pm}$ is a twisted version of equivariant K-theory,[1] and this paper can be seen as a preliminary step towards these other more elaborate versions.

2 The first definition

Given a space X with involution we have two natural K-theories, namely $K(X)$ and $K_{Z_2}(X)$ – the ordinary and equivariant theories respectively. Moreover we have the obvious homomorphism

$$\phi : K_{Z_2}(X) \to K(X) \tag{2.1}$$

[1] It is the twisting of equivariant K-theory by the non-trivial element of $H^1_{Z_2}(\text{pt}) = Z_2$. From the point of view of the equivariant graded Brauer group, $K_{\pm}(X)$ is the K-theory of the graded cross product algebra $C(X) \otimes M \rtimes Z_2$, where $C(X)$ is the algebra of continuous functions on X, and M is the graded algebra of 2×2-matrices over the complex numbers, graded in such a way that (i, j) entry has degree $i + j$. The action of Z_2 is the combination of the geometric action given on X and conjugation by the permutation matrix on M.

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which 'forgets' about the Z_2-action. We can reformulate this by introducing the space $(X \times Z_2)$ with the involution $(x, 0) \to (\tau(x), 1)$. Since this action is free we have

$$K_{Z_2}(X \times Z_2) \cong K(X)$$

and (2.1) can then be viewed as the natural homomorphism for K_{Z_2} induced by the projection

$$\pi : X \times Z_2 \to X. \tag{2.2}$$

Now, whenever we have such a homomorphism, it is part of a long exact sequence (of period 2) which we can write as an exact triangle

$$\begin{array}{ccc} K^*_{Z_2}(X) & \overset{\phi}{\rightarrow} & K^*(X) \\ \nwarrow & & \swarrow \delta \\ & K^*_{Z_2}(\pi) & \end{array} \tag{2.3}$$

where $K^* = K^0 \oplus K^1$, δ has degree 1 mod 2 and the relative group $K^*_{Z_2}(\pi)$ is just the relative group for a pair, when we replace π by a Z_2-homotopically equivalent inclusion. In this case a natural way to do this is to replace the X factor on the right of (2.2) by $X \times I$ where $I = [0, 1]$ is the unit interval with τ being reflection about the mid-point $\frac{1}{2}$. Thus, explicitly

$$K^*_{Z_2}(\pi) = K^*_{Z_2}(X \times I, X \times \partial I) \tag{2.4}$$

where ∂I is the (2-point) boundary of I.

We now take the group in (2.4) (with the degree shifted by one) as our definition of $K^*_\pm(X)$. It is then convenient to follow the notation of [1] where $R^{p,q} = R^p \oplus R^q$ with the involution changing the sign of the first factor, and we use K-theory with compact supports (so as to avoid always writing the boundary). Then our definition of $K_\pm$ becomes

$$K^0_\pm(X) = K^1_{Z_2}(X \times R^{1,0}) \cong K^0_{Z_2}(X \times R^{1,1}) \tag{2.5}$$

(and similarly for K^1).

Let us now explain why this definition fits the geometric situation we began with (and which comes from the physics). Given a vector bundle E we can form the pair $(E, \tau^* E)$ or the virtual bundle

$$E - \tau^* E.$$

Under the involution, E and $\tau^* E$ switch and the virtual bundle goes into its negative. Clearly, if E came from an equivariant bundle, then $E \cong \tau^* E$ and the virtual bundle is zero. Hence the virtual bundle depends only the element

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defined by E in the cokernel of ϕ, and hence by the image of E in the next term of the exact sequence (2.3), i.e. by

$$\delta(E) \in K^0_{\pm}(X).$$

This explains the link with our starting point and it also shows that one cannot always define $K_{\pm}(X)$ in terms of such virtual bundles on X. In general the exact sequence (2.3) does not break up into short exact sequences and δ is not surjective.

At this point a physicist might wonder whether the definition of $K_{\pm}(X)$ that we have given is the right one. Perhaps there is another group which is represented by virtual bundles. We will give two pieces of evidence in favour of our definition, the first pragmatic and the second more philosophical.

First let us consider the case when the involution τ on X is trivial. Then $K^*_{Z_2}(X) = R(Z_2) \otimes K^*(X)$ and $R(Z_2) = Z \oplus Z$ is the representation ring of Z_2 and is generated by the two representations:

$$\begin{array}{ll} 1 & \text{(trivial representation)} \\ \rho & \text{(sign representation).} \end{array}$$

The homomorphism ϕ is surjective with kernel $(1-\rho)K^*(X)$ so $\delta = 0$ and

$$K^0_{\pm}(X) \cong K^1(X). \tag{2.6}$$

This fits with the requirements of the physics, which involves a switch from type IIA to type IIB string theory. Note also that it gives an extreme example when ∂ is not surjective.

Our second argument is concerned with the general passage from physical (quantum) theories to topology. If we have a theory with some symmetry then we can consider the quotient theory, on factoring out the symmetry. Invariant states of the original theory become states of the quotient theory but there may also be new states that have to be added. For example if we have a group G of geometric symmetries, then closed strings in the quotient theory include strings that begin at a point x and end at $g(x)$ for $g \in G$. All this is similar to what happens in topology with (generalized) cohomology theories, such as K-theory. If we have a morphism of theories, such as ϕ in (2.1) then the third theory we get fits into a long-exact sequence. The part coming from $K(X)$ is only part of the answer – other elements have to be added. In ordinary cohomology where we start with cochain complexes the process of forming a quotient theory involves an ordinary quotient (or short exact sequence) at the level of cochains. But this becomes a long exact sequence at the cohomology level. For K-theory the analogue is to start with bundles over small open sets and at this level we can form the naïve quotients, but the K-groups arise when

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we impose the matching conditions to get bundles, and then we end up with long exact sequences.

It is also instructive to consider the special case when the involution is free so that we have a double covering $\tilde{X} \to X$ and the exact triangle (2.3), with $\tilde{X}$ for X, becomes the exact triangle

$$\begin{array}{ccc} K^*(X) & \overset{\phi}{\to} & K^*(\tilde{X}) \\ \nwarrow & & \swarrow \delta \\ & K^*_{Z_2}(L) & \end{array} \tag{2.7}$$

Here L is the real line bundle over X associated to the double covering $\tilde{X}$ (or to the corresponding element of $H^1(X, Z_2)$), and we again use compact supports. Thus (for $q = 0, 1 \mod 2$)

$$K^q_{\pm}(\tilde{X}) = K^{q+1}(L). \tag{2.8}$$

If we had repeated this argument using equivariant cohomology instead of equivariant K-theory we would have ended up with the twisted cohomology mentioned earlier, via a twisted suspension isomorphism

$$H^q(X, \tilde{Z}) = H^{q+1}(L). \tag{2.9}$$

This shows that, for free involutions, $K_{\pm}$ **is precisely the analogue of twisted cohomology,** so that, for example, the Chern character of the former takes values in the rational extension of the latter.

3 Relation to Fredholm operators

In this section we shall give another definition of $K_{\pm}$ which ties in naturally with the analysis of Fredholm operators, and we shall show that this definition is equivalent to the one given in Section 2.

We begin by recalling a few basic facts about Fredholm operators. Let H be complex Hilbert space, $\mathcal{B}$ the space of bounded operators with the norm topology and $\mathcal{F} \subset \mathcal{B}$ the open subspace of Fredholm operators, i.e. operators A so that $\ker A$ and $\operatorname{coker} A$ are both finite-dimensional. The index defined by

$$\operatorname{index} A = \dim \ker A - \dim \operatorname{coker} A$$

is then constant on connected components of $\mathcal{F}$. If we introduce the adjoint A^* of A then

$$\operatorname{coker} A = \ker A^*$$

so that

$$\operatorname{index} A = \dim \ker A - \dim \ker A^*.$$

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More generally if we have a continuous map

$$f : X \to \mathfrak{F}$$

(i.e. a family of Fredholm operators, parametrized by X), then one can define

$$\text{index } f \in K(X)$$

and one can show [2] that we have an isomorphism

$$\text{index} : [X, \mathfrak{F}] \cong K(X) \tag{3.1}$$

where [,] denotes homotopy classes of maps. Thus $K(X)$ has a natural definition as the 'home' of indices of Fredholm operators (parametrized by X): it gives the complete homotopy invariant.

Different variants of K-theory can be defined by different variants of (3.1). For example real K-theory uses real Hilbert space and equivariant K-theory for G-spaces uses a suitable H-space module of G, namely $L_2(G) \otimes H$. It is natural to look for a similar story for our new groups $K_\pm(X)$. A first candidate might be to consider Z_2-equivariant maps

$$f : X \to \mathfrak{F}$$

where we endow $\mathfrak{F}$ with the involution $A \to A^*$ given by taking the adjoint operator. Since this switches the role of kernel and cokernel it acts as -1 on the index, and so is in keeping with our starting point.

As a check we can consider X with a trivial involution, then f becomes a map

$$f : X \to \widehat{\mathfrak{F}}$$

where $\widehat{\mathfrak{F}}$ is the space of self-adjoint Fredholm operators. Now in [4] it is shown that $\widehat{\mathfrak{F}}$ has three components

$$\widehat{\mathfrak{F}}_+, \widehat{\mathfrak{F}}_-, \widehat{\mathfrak{F}}_*$$

where the first consists of A which are essentially positive (only finitely many negative eigenvalues), the second is given by essentially negative operators. These two components are trivial, in the sense that they are contractible, but the third one is interesting and in fact [4]

$$\widehat{\mathfrak{F}}_* \sim \Omega\mathfrak{F} \tag{3.2}$$

where Ω denotes the loop space. Since

$$[X, \Omega\mathfrak{F}] \cong K^1(X)$$

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this is in agreement with (2.6) – though to get this we have to discard the two trivial components of $\widehat{\mathcal{F}}$, a technicality to which we now turn.

Lying behind the isomorphism (3.1) is Kuiper's Theorem [11] on the contractibility of the unitary group of Hilbert spaces. Hence to establish that our putative definition of $K_\pm$ coincides with the definition given in Section 2 we should expect to need a generalization of Kuiper's Theorem incorporating the involution $A \to A^*$ on operators. The obvious extension turns out to be false, precisely because $\widehat{\mathcal{F}}$, the fixed-point set of $*$ on $\mathcal{F}$, has the additional contractible components. There are various ways we can get round this but the simplest and most natural is to use 'stabilization'. Since $H \cong H \oplus H$ we can always stabilize by adding an additional factor of H. In fact Kuiper's Theorem has two parts in its proof:

(1) The inclusion $U(H) \to U(H \oplus H)$ defined by $u \to u \oplus 1$ is homotopic to the constant map.
(2) This inclusion is homotopic to the identity map given by the isomorphism $H \cong H \oplus H$.

The proof of (1) is an older argument (sometimes called the 'Eilenberg swindle'), based on a correct use of the fallacious formula

$$\begin{aligned} 1 &= 1 + (-1 + 1) + (-1 + 1) \ldots \\ &= (1 + -1) + (1 + -1) + \ldots \\ &= 0. \end{aligned}$$

The trickier part, and Kuiper's contribution, is the proof of (2).

For many purposes, as in K-theory, the stronger version is a luxury and one can get by with the weaker version (1), which applies rather more generally. In particular (1) is consistent with taking adjoints (i.e. inverses in $U(H)$), which is the case we need.

With this background explanation we now introduce formally our second definition, and to distinguish it temporarily from $K_\pm$ as defined in Section 2, we put

$$\mathcal{K}_\pm(X) = [X, \mathcal{F}]^s_* \tag{3.3}$$

where $*$ means we use Z_2-maps compatible with $*$ and s means that we use **stable** homotopy equivalence. More precisely the Z_2-maps

$$f : X \to \mathcal{F}(H) \qquad g : X \to \mathcal{F}(H)$$

are called stably homotopic if the 'stabilized' maps

$$f^s : X \to \mathcal{F}(H \oplus H) \qquad g^s : X \to \mathcal{F}(H \oplus H)$$

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