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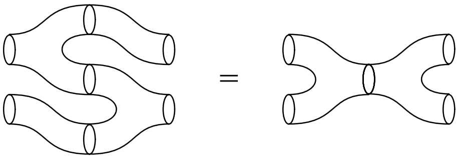

FRONTISPIECE *The topological expression of the main axiom of a Frobenius algebra.* This figure illustrates the content of the main theorem, and captures the whole spirit of the book.

Frobenius Algebras and 2D Topological Quantum Field Theories

JOACHIM KOCK

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For fun

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Preface

The text centres around notions of Frobenius structure which in recent years have drawn some attention in topology, physics, algebra, and computer science. In topology the structure arises in the category of 2-dimensional oriented cobordisms (and their linear representations, which are 2-dimensional topological quantum field theories) – this is the subject of the first chapter. The main result here (due to Abrams [1]) is a description in terms of generators and relations of the monoidal category ***2Cob***. In algebra, the structure manifests itself simply as Frobenius algebras, which are treated carefully in Chapter 2. The main result here is a characterisation of Frobenius algebras in terms of comultiplication which goes back to Lawvere [32] and was rediscovered by Quinn [43] and Abrams [1]. The main result of these notes is that these two categories are equivalent: the category of 2-dimensional topological quantum field theories and the category of commutative Frobenius algebras. This result is due to Dijkgraaf [16], further details of the proof having been provided by Quinn [43], Dubrovin [19], and Abrams [1]. The notions from category theory needed in order to express this rigorously (monoidal categories and their linear representations) are developed from an elementary level in Chapter 3. The categorical viewpoint allows us to extract the essence of what is going on in the first two chapters, and prove a natural generalisation of the theorem. To arrive at this insight, we carefully review the classical fact that the simplex category Δ is the free monoidal category on a monoid. (This means in particular that there is an equivalence of categories between the category of algebras and the category of 'linear representations' of Δ.) Now the notion of a Frobenius object in a monoidal category is introduced, and the promised generalisation of the theorem (main result of Chapter 3) states that ***2Cob*** is the free symmetric monoidal category on a commutative Frobenius object.

For more details on the mathematical content, see the Introduction.

The target. The book is based on notes prepared for an intensive two-week mini-course for advanced undergraduate students, given in the UFPE Summer School, Recife, Brazil, in January 2002. The prerequisites are modest: the students of the mini-course were expected to have followed these three standard courses taught at Brazilian universities: one on *differential topology*, one on *algebraic structures* (groups and rings) and one *second course in linear algebra*. From topology we need just some familiarity with the basic notions of differentiable manifolds; from algebra we need basic notions of rings and ideals, groups and algebras; and first and foremost the reader is expected to be familiar with tensor products and hom sets. Usually the course *algebraic structures* contains an introduction to categories and functors, but not enough to get acquainted with the categorical way of thinking and appreciate it; the exposition in this text is meant to take this into account. The basic definitions are given in an appendix, and the more specialised notions are introduced with patience and details, and with many examples – and hopefully the interplay between topology and algebra will provide the appreciation of the categorical viewpoint.

In a wider context these notes are targeted at undergraduate students with a similar background, as well as graduate students of all areas of mathematics. Experienced mathematicians and experts in the field will sometimes be bored by the amount of detail presented, but it is my hope the drawings will keep them awake.

The aim. At an immediate level, the aim of these notes is simply to expose some delightful and not very well known mathematics where a lot of figures can be drawn: a quite elementary and very nice interaction between topology and algebra – and rather different in flavour from what one learns in a course in algebraic topology. On a deeper level, the aim is to convey an impression of unity in mathematics, an aspect which is often hidden from students until later in their mathematical apprenticeship. Finally, perhaps the most important aim is to use this as motivation for category theory, and specifically to serve as an introduction to monoidal categories.

Admittedly, the main theorem is not a particularly useful tool that the students will draw upon again and again throughout their mathematical career, and one could argue that the time would be better spent on a course on group representations or distributions, for instance. But after all, this is a summer school (and this is Brazil!): maximising the throughput is not our main concern – the wonderful relaxed atmosphere I know from previous summer schools in Recife is much more important – I hope the students when they go to the beach in the weekend will make drawings of 2-dimensional cobordisms in the sand! (I think they would not take orthogonality relations or Fourier transforms with them to the beach. . .)

What the lectures are meant to give the students are rather some techniques and viewpoints, and in the end this categorical perspective reduces the main theorem to a special case of general principles. A lot of emphasis is placed on universal properties, symmetry, distinction between structure and property, distinction between identity and natural isomorphism, the interplay between graphical and algebraic approaches to mathematics – as well as reflection on the nature of the most basic operations of mathematics: multiplication and addition. Getting acquainted with such categorical viewpoints in mathematics is certainly a good investment.

Finally, to be more concrete, the techniques learned in this course should constitute a good primer for going into quantum groups or knot theory.

The source – acknowledgements. The idea of these notes originated in a workshop I led at KTH, Stockholm, in 2000, whose first part was devoted to understanding the paper of Abrams [1] (corresponding more or less to Chapters 1 and 2 of this text). I am thankful for the contributions of the core participants of the workshop: ***Carel*** Faber, ***Helge*** Måkestad, ***Mats*** Boij, and ***Michael*** Shapiro, and in particular to ***Dan*** Laksov, for many fruitful discussions about Frobenius algebras.

The more categorical viewpoint of Chapter 3 was influenced by the people I work with here in Nice; I am indebted in particular to ***André*** Hirschowitz and ***Bertrand*** Toën. I have also benefited from discussions and email correspondence with ***Arnfinn*** Laudal, ***Göran*** Fors, ***Jan*** Gorski, ***Jean-Louis*** Cathélineau, ***John*** Baez, and ***Pedro*** Ontaneda, all of whom are thanked. I am particularly indebted to ***Anders*** Kock, ***Peter*** Johnson, and ***Tom*** Leinster for many discussions and helpful emails, and for carefully reading preliminary versions of the manuscript, pointing out grim errors, annoying inaccuracies, and misprints.

Israel Vainsencher, ***Joaquim*** Roé, ***Ramón*** Mendoza, and ***Sérgio*** Santa Cruz also picked up some misprints – thanks. My big sorrow about these notes is that I do not understand the physics behind it all, in spite of a great effort by ***José*** Mourão to explain it to me – I am grateful to him for his patience.

During the redaction of these notes I have reminisced about maths classes in primary school, and some of the figures are copied from my very first maths books. Let me take the opportunity to thank ***Marion*** Kuhlmann and ***Jørgen*** Skaftved for the mathematics they taught me when I was a child.

During my work with this subject and specifically with these notes, I have been supported by ***The National Science Research Council of Denmark***, The Nordic Science Research Training Academy ***NorFA***, and (currently) a Marie Curie Fellowship from ***The European Commission***. In neither case was I supposed to spend so much time with Frobenius algebras and topological quantum field theories – it is my hope that these notes, as a concrete outcome of the time spent, do it justice to some extent.

I am indebted to my wife ***Andrea*** for her patience and support.

Last but not least, I wish to thank the organisers of the Summer School in Recife – in particular ***Letterio*** Gatto – for inviting me to give this mini-course, which in addition to being a very dear opportunity to come back to Recife – *Voltei, Recife! foi a saudade que me trouxe pelo braço* – has also been a welcome incentive to work out the details of this material and learn a lot of mathematics.

Feedback is most welcome. Please point out mathematical errors or misunderstandings, misleading viewpoints, unnecessary pedantry, or things that should be better explained; typos, mispellings, bad English, TEX-related issues. I intend to keep a list of errata on my web site.

The original LATEX source files were prepared in alpha. The figures were coded with the texdraw package, written by Peter Kabal. The diagrams were set using the diagrams package of Paul Taylor, except for the curved arrows which were coded by hand.

Contents

General conventions

We consistently write composition of functions (or arrows) from the left to the right: given functions (or arrows)

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

we denote the composite *fg*. Similarly, we put the symbol of a function to the right of its argument, writing for example

$$\begin{aligned} f : X &\longrightarrow Y \\ x &\longmapsto xf. \end{aligned}$$