Part I

Hyperbolic 3-manifolds

Kleinian Groups and Hyperbolic 3-Manifolds Y. Komori, V. Markovic & C. Series (Eds.)
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Combinatorial and geometrical aspects of hyperbolic 3-manifolds

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1. Introduction

This is the edited and revised form of handwritten notes that were distributed with the lectures that I gave at the conference in Warwick on September 11–15 of 2001.[2] The goal of the lectures was to expose some recent work [Min03] on the structure of ends of hyperbolic 3-manifolds, which is part of a program to solve Thurston's Ending Lamination Conjecture (the conclusion of the program, which is joint work with J. Brock and R. Canary, will appear in [BCM]). In the interests of simplicity and the ability to get to the heart of the matter, the notes are quite informal in their treatment of background material, and the main results are often stated in special cases, with detailed examples taking the place of proofs. Thus it is hoped that the reader will be able to extract the main ideas with a minimal investment of effort, and in the event he or she is still interested, can obtain the details in [Min03], which will appear later on.

I would like to thank the organizers of the conference for inviting me and giving me the opportunity to talk for what must have seemed like a very long time.

1.1. Object of Study

If the interior N of a compact 3-manifold $\overline{N}$ admits a complete infinite-volume hyperbolic structure, then there is a multidimensional *deformation space* of such structures. The study of this space goes back to Poincaré and Klein, but the modern theory began with Ahlfors-Bers in the 1960's and received the perspective that we will focus on from Thurston and others in the late 70's. The deformation theory depends deeply on an understanding of the geometry of the *ends* of N (in the sense of Freudenthal [Fre42]), which one can think of as small neighborhoods of the boundary components of $\overline{N}$.

[1]Based on work partially supported by NSF grant DMS-9971596.

[2]The terrible events in New York that coincided with the beginning of this conference overshadow its subject matter in significance, and yet those same events demand of us to continue with our ordinary work.

The interior of the deformation space, as studied by Ahlfors, Bers [AB60, Ber60, Ber70b], Kra [Kra72], Marden [MMa79, Mar74], Maskit [Mak75], and Sullivan [Sul85], can be parametrized using the Teichmüller space of $\partial\overline{N}$ – that is, by choosing a "conformal structure at infinity" for each (non-toroidal) boundary component of $\overline{N}$. (See also [KS93] and [BO01] for other approaches to the study of the interior). The boundary contains manifolds with parabolic cusps [Mak70, McM91], and more generally, with *geometrically infinite ends* [Ber70a, Gre66, Thu79]. The Teichmüller parameter is replaced by Thurston's *ending laminations* for such ends. Thurston conjectured [Thu82] that these invariants are sufficient to determine the geometry of N uniquely – this is known as the Ending Lamination Conjecture (see also [Abi88] for a survey).

In these notes we will consider the special case of *Kleinian surface groups*, for which $\pi_1(N)$ is isomorphic to $\pi_1(S)$ for a surface S. This case suffices for describing the ends of general N, provided $\partial\overline{N}$ is incompressible. (In the compressible case the deeper question of Marden's *tameness conjecture* comes in, and this is beyond the scope of our discussion. See Marden [Mar74] and Canary [Can93b].)

We will show how the the ending laminations, together with the combinatorial structure of the set of simple closed curves on a surface, allows us to build a *Lipschitz model* for the geometric structure of N, which in particular describes the thick-thin decomposition of N. These results, which are proven in detail in [Min03], will later be followed by *bilipschitz* estimates in Brock–Canary–Minsky [BCM], and these will suffice to prove Thurston's conjecture in the case of incompressible boundary.

1.2. Kleinian surface groups

From now on, let S be an oriented compact surface with $\chi(S) < 0$, and let

$$\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$$

be a discrete, faithful representation. If $\partial S \neq \emptyset$ we require $\rho(\gamma)$ to be parabolic for γ representing any boundary component. This is known as a (marked) Kleinian surface group. We name the quotient 3-manifold

$$N = N_\rho = \mathbb{H}^3/\rho(\pi_1(S)).$$

Periodic manifolds Before discussing the general situation let us consider a well-known and especially tractable example.

Let $\varphi : S \to S$ be a pseudo-Anosov homeomorphism (this means that φ leaves no finite set of non-boundary curves invariant up to isotopy). The mapping torus of φ is

$$M_\varphi = S \times \mathbb{R}/\langle (x,t) \mapsto (\varphi(x), t+1)\rangle,$$

a surface bundle over S^1 with fibre S and monodromy φ. Thurston [Thu86b] showed, as part of his hyperbolization theorem, that $int(M_\varphi)$ admits a hyperbolic structure which we'll call N_φ (see also Otal [Ota96] and McMullen [McM96]). Let $N \cong int(S) \times \mathbb{R}$ be the infinite cyclic cover of N_φ, "unwrapping" the circle direction (Figure 1). After identifying S with some lift of the fibre, we obtain an isomorphism $\rho : \pi_1(S) \to \pi_1(N) \subset \mathrm{PSL}_2(\mathbb{C})$, which is a Kleinian surface group.

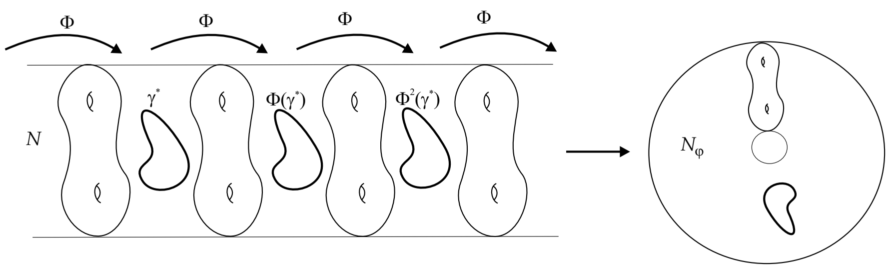

Figure 1: N covers the surface bundle N_φ.

The deck translation $\Phi : N \to N$ of the covering induces $\Phi_* = \varphi_* : \pi_1(S) \to \pi_1(S)$. We next consider the action of φ on the space of *projective measured laminations* $\mathscr{PML}(S)$ (see [FLP79, Bon01], and Lecture 3). For every simple closed curve γ in S, the sequences $[\varphi^n(\gamma)]$ and $[\varphi^{-n}(\gamma)]$ converge to two distinct points ν_+ and ν_- in $\mathscr{PML}(S)$. After isotopy, φ can be represented on S by a map that preserves the leaves of both ν_+ and ν_-, stretching the former and contracting the latter.

We can see $\nu_\pm$ directly in the asymptotic geometry of N: For a curve γ in S, let γ^* be its geodesic representative in N. Now consider $\Phi^n(\gamma^*)$ – these are all geodesics of the same length, marching off to infinity in both directions as $n \to \pm\infty$, and note that $\Phi^n(\gamma^*) = \varphi^n(\gamma)^*$. So, we have a sequence of simple curves in S, converging to ν_+ as $n \to \infty$, whose geodesic representatives "exit the + end" of N (similarly as $n \to \infty$ they converge to ν_- and the geodesics exit the other end).

The laminations $\nu_\pm$ are the ending laminations of ρ in this case. To understand the general case we will have to develop a bit of terminology, and recall the work of Thurston and Bonahon.

Ends Let N_0 denote N minus its cusps (each cusp is an open solid torus, whose boundary in N is a properly embedded open annulus). The relative version of Scott's core theorem (see McCullough [McC86], Kulkarni–Shalen [KSh89] and Scott [Sco73b]) gives us a compact submanifold K in N_0, homeomorphic to $S \times [0,1]$, which meets each cusp boundary in an annulus (including the annuli $\partial S \times [0,1]$). The components of $N_0 \setminus K$ are in one to one correspondence with the topological ends of N_0, and are called neighborhoods of the ends (see Bonahon [Bon86]).

N also has a *convex core* C_N, which is the smallest closed convex submanifold whose inclusion is a homotopy equivalence. Each end neighborhood either meets C_N in a bounded set, in which case the end is called *geometrically finite*, or is contained in C_N, in which case the end is *geometrically infinite*.

From now on, let us assume that N has *no extra cusps*, which means that the cusps correspond only to the components of ∂S. In particular N_0 has exactly two ends, which we label $+$ and $-$ according to an appropriate convention.

Simply degenerate ends In [Thu79], Thurston made the following definition, which can be motivated by the surface bundle example:

Definition 1.1. An end of N is **simply degenerate** if there exists a sequence of simple closed curves α_i in S such that α_i^* exit the end.

Here "exiting the end" means that the geodesics are eventually contained in an arbitrarily small neighborhood of the end, and in particular outside any compact set. Note that a geometrically finite end cannot be simply degenerate, since all closed geodesics are contained in the convex hull.

Thurston then established this theorem (stated in the case without extra cusps):

Theorem 1.2. [Thu79] *If an end e of N is simply degenerate then there exists a unique lamination ν_e in S such that for any sequence of simple closed curves α_i in S,*

$$\alpha_i \to \nu_e \iff \alpha_i^* \text{ exit the end } e.$$

A sequence $\alpha_i \to \nu_e$ can be chosen so that the lengths $\ell_N(\alpha_i^*) \leq L_0$, where L_0 depends only on S.

Furthermore, ν_e fills S – its complement consists of ideal polygons and once-punctured ideal polygons.

(We are being cagey here about just what kind of lamination ν_e is, and what convergence $\alpha_i \to \nu_e$ means. See Lecture 3 for more details.)

Thurston also proved that simply degenerate ends are *tame*, meaning that they have neighborhoods homeomorphic to $S \times (0, \infty)$, and that manifolds obtained as limits of quasifuchsian manifolds have ends that are geometrically finite or simply degenerate. Bonahon completed the picture with his "tameness theorem",

Theorem 1.3. [Bon86] *The ends of N are either geometrically finite or simply degenerate.*

In particular N_0 is homeomorphic to $S \times \mathbb{R}$, and ending laminations are well-defined for each geometrically infinite end.

Geometrically finite ends are the ones treated by Ahlfors, Bers and their coworkers, and their analysis requires a discussion of quasiconformal mappings and Teichmüller theory (see [Ber60, Ber70b, Sul86] for more). In order to simplify our exposition we will limit ourselves, for the remainder of these notes, to Kleinian surface groups ρ with no extra cusps, and with no geometrically finite ends. In particular the convex hull of N_ρ is all of N_ρ, and there are two ending laminations, ν_+ and ν_-. This is called the *doubly degenerate* case.

1.3. Models and bounds

Our goal now is to recover geometric information about N_ρ from the asymptotic data encoded in $\nu_\pm$. The following natural questions arise, for example:

- Thurston's Theorem 1.2 guarantees the existence of a sequence $\alpha_i \to \nu_+$ whose geodesic representatives have bounded lengths $\ell_N(\alpha_i^*)$. How can we determine, from ν_+, which sequences have this property?
- The case of the cyclic cover of a surface bundle is not typical: because it covers a compact manifold (except for cusps), it has "bounded geometry". That is,

  $$\inf_\beta \ell_N(\beta) > 0$$

  where β varies over closed geodesics. The bounded geometry case is considerably easier to understand. In particular the Ending Lamination Conjecture in this category (without cusps) was proven in [Min93, Min94].

  Can we tell from $\nu_\pm$ alone whether N has bounded geometry?
- If N doesn't have bounded geometry, there are arbitrarily short closed geodesics in N, each one encased in a *Margulis tube*, which is a standard collar neighborhood. Such examples were shown to exist by Thurston [Thu86b] and Bonahon-Otal [BO88], and to be generic in an appropriate sense by McMullen [McM91].

  In the unbounded geometry case, can we tell *which* curves in N are short? How are they arranged in N?

We will describe the construction of a "model manifold" M_ν for N, which can be used to answer these questions. M_ν is constructed combinatorially from $\nu_\pm$, and

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contains for example solid tori that correspond to the Margulis tubes of short curves in N. M_ν comes equipped with a map

$$f : M_\nu \to N$$

which takes the solid tori to the Margulis tubes, is proper, Lipschitz in the complement of the solid tori, and preserves the end structure. This will be the content of the Lipschitz Model Theorem, which will be stated precisely in Lecture 6.

Note that if f is *bilipschitz* then the Ending Lamination Conjecture follows: If N_1, N_2 have the same invariants $\nu_\pm$ then the same model M_ν would admit bilipschitz maps $f_1 : M_\nu \to N_1$ and $f_2 : M_\nu \to N_2$, and $f_2 \circ f_1^{-1} : N_1 \to N_2$ would be a bilipschitz homeomorphism. By Sullivan's Rigidity Theorem [Sul81a], N_1 and N_2 would be isometric.

1.4. Plan

Here is a rough outline of the remaining lectures:

§**2** **Hierarchies and model manifolds:** We will show how to build M_ν starting with a geodesic in the *complex of curves* $\mathscr{C}(S)$. The main tool is the *hierarchy of geodesics* developed in Masur-Minsky [MM00]. Much of the discussion will take place in the special case of the 5-holed sphere $S_{0,5}$, where the definitions and arguments are considerably simplified.

§**3** **From ending laminations to model manifold:** Using a theorem of Klarreich we will relate ending laminations to *points at infinity* for $\mathscr{C}(S)$, and this will allow us to associate to a pair of ending laminations a geodesic in $\mathscr{C}(S)$, and its associated hierarchy and model manifold.

§**4** **The quasiconvexity argument:** We then begin to explore the linkage between geometry of the 3-manifold N_ρ and the curve complex data. We will show that the subset of $\mathscr{C}(S)$ consisting of curves with bounded length in N is *quasiconvex*. The main tool here is an argument using pleated surfaces and Thurston's Uniform Injectivity Theorem.

§**5** **Quasiconvexity and projection bounds:** In this lecture we will discuss the Projection Bound Theorem, a strengthening of the Quasiconvexity Theorem that shows that curves that appear in the hierarchy are combinatorially close to the bounded-length curves in N. We will also prove the Tube Penetration Theorem, which controls how deeply certain pleated surfaces can enter into Margulis tubes.

§**6 A priori length bounds and model map:** Applying the Projection Bound Theorem and the Tube Penetration Theorem, we will establish a uniform bound on the lengths of all curves that appear in the hierarchy.

We will then state the Lipschitz Model Theorem, whose proof uses the a priori bound and a few additional geometric arguments. As consequences we will obtain some final statements on the structure of the set of short curves in N.

2. Curve complex and model manifold

In this lecture we will introduce the complex of curves $\mathscr{C}(S)$ and demonstrate how a geodesic in $\mathscr{C}(S)$ leads us to construct a "model manifold". For simplicity we will mostly work with $S = S_{0,5}$, the sphere with 5 holes. (In general let $S_{g,n}$ be the surface with genus g and n boundary components).

2.1. The complex of curves

$\mathscr{C}(S)$ will be a simplicial complex whose vertices are homotopy classes of simple, essential, unoriented closed curves ("Essential" means homotopically nontrivial, and not homotopic to the boundary). Barring the exceptions below, we define the k-simplices to be unordered $k+1$-tuples $[v_0 \dots v_k]$ such that $\{v_i\}$ can be realized as pairwise disjoint curves. This definition was given by Harvey [Hav81].

Exceptions: If $S = S_{0,4}$, $S_{1,0}$ or $S_{1,1}$ then this definition gives no edges. Instead we allow edges $[vw]$ whenever v and w can be realized with

$$\#v \cap w = \begin{cases} 1 & S_{1,0}, S_{1,1} \\ 2 & S_{0,4}. \end{cases}$$

(see Figure 2). In this case $\mathscr{C}(S)$ is the *Farey graph* in the plane: a vertex is indexed by the slope p/q of its lift to the planar $\mathbb{Z}^2$ cover of S, so the vertex set is $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \infty$. Two vertices p/q, r/s are joined by an edge if $|ps - qr| = 1$ (see e.g. Series [Ser85a] or [Min99]).

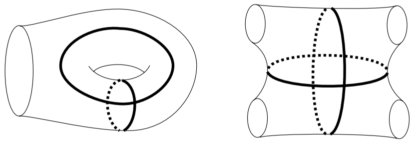

Figure 2: Adjacent vertices in $\mathscr{C}(S_{1,1})$ and $\mathscr{C}(S_{0,4})$

For $S_{0,0}, S_{0,1}, S_{0,2}, S_{0,3}$: $\mathscr{C}(S)$ is empty. (For the annulus $S_{0,2}$ there is another useful construction which we will return to later.)

Let $\mathscr{C}_k(S)$ denote the k-skeleton of $\mathscr{C}(S)$. We will concentrate on $\mathscr{C}_0$ and $\mathscr{C}_1$.

We endow $\mathscr{C}(S)$ with the metric that makes every simplex regular Euclidean of sidelength 1. Thus $\mathscr{C}_1(S)$ is a graph with unit-length edges. Consider a geodesic in $\mathscr{C}_1(S)$ – it is a sequence of vertices $\{v_i\}$ connected by edges (Figure 3), and in particular: v_i, v_{i+1} are disjoint (in the non-exceptional cases), v_i and v_{i+2} intersect but are disjoint from v_{i+1}, and v_i and v_{i+3} *fill* the surface: their union intersects every essential curve. It is harder to characterize topologically the relation between v_i and v_j for $j > i+3$.

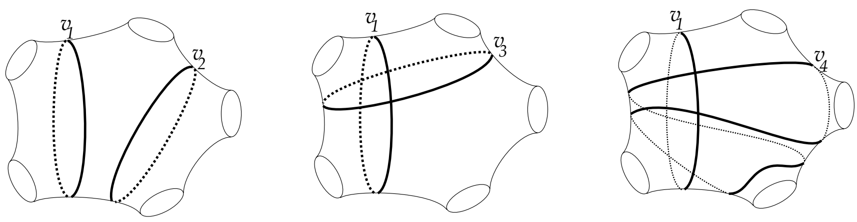

Figure 3: $\{v_1, v_2, v_3, v_4\}$ are the vertices of a geodesic in $\mathscr{C}(S_{0,5})$.

2.2. Model construction

Let $S = S_{0,5}$ – this case is considerably simpler than the general case, while preserving many of the main features.

Starting with a *bi-infinite geodesic* g in $\mathscr{C}_1(S)$ (more about the existence of such geodesics later), we will construct a manifold $M_g \cong S \times \mathbb{R}$, equipped with a piecewise-Riemannian metric. M_g is made of "standard blocks", all isometric, and "tubes", or solid tori of the form (annulus) × (interval).

Hierarchy We begin by "thickening" g in the following sense: Any vertex $v \in \mathscr{C}_0(S)$ divides S into two components, one $S_{0,3}$ and one $S_{0,4}$. Let W_v denote the second of these. If v_i is a vertex of g then

$$v_{i-1}, v_{i+1} \in \mathscr{C}_0(W_{v_i}).$$

The complex $\mathscr{C}(W_{v_i})$ is just the Farey graph, and we may join v_{i-1} to v_{i+1} by a geodesic in that graph. Name this geodesic h_i, and represent it schematically as in Figure 4.

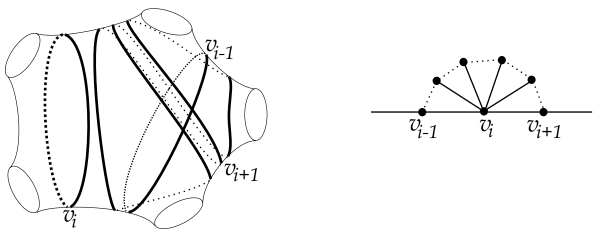

Figure 4: The local configuration at a vertex v_i of g yields a "wheel" in the link of v_i. Note, edges of h_i are not edges of $\mathscr{C}(S)$; call them "rim" edges. The other edges are called "spokes".

We repeat this at every vertex. The resulting system is called a *hierarchy of geodesics*. (In general surfaces, considerable complications arise. Geodesics must satisfy a technical condition called "tightness", and the hierarchy has more levels. This is joint work with Masur [MM99, MM00].)

Note that the construction is not uniquely dependent on g – geodesics are not always unique in the Farey graph, so there are arbitrary choices for each h_i. However what we have to say will work regardless of how the choices are made.

Blocks To each rim edge e we associate a "block" $B(e)$, and then glue these together to form the model manifold. e is an edge of $\mathscr{C}(W_v)$ for some vertex v – denote $W_e \equiv W_v$ for convenience. Let e^-, e^+ be its vertices, ordered from left to right. Let C_+ and C_- be open collar neighborhoods of e^+ and e^-, respectively. We define

$$B(e) = W_e \times [-1,1] - (C_+ \times (1/2,1] \cup C_- \times [-1,-1/2)).$$

Thus we have removed solid-torus "trenches" from the top and bottom of the product $W_e \times [-1,1]$. Figure 5 depicts this as a gluing construction.

The boundary $\partial B(e)$ divides into four *3-holed spheres*,

$$\partial_\pm B(e) \equiv (W_e - C_\pm) \times \pm 1$$

and some *annuli*. Schematically, we depict this structure in Figure 6.

Gluing Take the disjoint union of all the blocks arising from the hierarchy over g, and glue them along 3-holed sphere, where possible. That is, if $Y \times \{1\}$ appears in $\partial_+ B(e_1)$ and $Y \times \{-1\}$ appears in $\partial_- B(e_2)$, identify them using the identity map in Y.

(A technicality we are eliding is that subsurfaces are determined only up to isotopy; one can select one representative for each isotopy class in a fairly nice and consistent way.)

There are three types of gluings that can occur: