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W.T. Tutte 1917-2002

Norman Biggs

Bill Tutte will take his place in history for two reasons. First, as part of the now-famous codebreaking team at Bletchley Park, he made a significant individual contribution to the outcome of the Second World War. Secondly, he formulated and proved many of the theorems that form the foundations of Graph Theory. In both cases his achievements resulted from very deep insights into matters that, at first sight, might be thought simple. In 1977 Paul Erdős recalled [13]:

> I first heard about Tutte in early September 1939 .... But of the real powers I only learned later. T. Gallai and I as freshmen in 1930 took the course on graph theory by D. Kőnig – he mentioned the conjecture of Tait and the extension of Petersen's theorem on factorisation of graphs as important outstanding problems – we tried both unsuccessfully. As is well known, Tutte settled both problems – and many others.

William Thomas Tutte was born on May 14, 1917 at Fitzroy House, a horseracing establishment in Newmarket, England. Around 1921 the family settled in Chevely, a village near Newmarket, where his father was gardener at the Rutland Arms Hotel. He attended the village school until, at the age of 11, he won a scholarship to the Cambridge and County High School. The school was 15 miles from his home and the daily journey was difficult, but well worth the effort. He won many prizes, and in the school library he discovered W.W. Rouse Ball's book of *Mathematical Recreations and Essays*, in which he read about the Five-Colour Theorem and Petersen's Theorem. Both these results were to figure largely in his life's work.

In 1935 he went up to Trinity College, Cambridge, supported by a State Scholarship and a College Scholarship. He read Natural Sciences, specialising in Chemistry, and getting a First Class Honours degree. He also joined the Trinity Mathematical Society, where he met R.L. Brooks, C.A.B. Smith, and A.H. Stone. Together they worked on the problem of dividing a square into squares of different sizes. The story of their work on this problem has been told many times, including by Tutte himself in the *Scientific American* [6]. It is always a surprise to find just how many important ideas arose first in this work on a problem in 'recreational' mathematics. The ideas about spanning trees can be traced back to Kirchhoff, but many of their algebraic results were new, as were the insights into planarity and duality.

By the time that the paper on squaring the square was published [1], Tutte had started his research in Chemistry, and had produced two papers on his experimental results. His progress was interrupted when he was called up for national service in the Second World War, and after initial training he arrived at Bletchley Park, the British cryptographic HQ, in 1941. He was one of many who regarded signing the Official Secrets Act as a lifelong obligation, and when stories of the great deeds done at Bletchley began to leak out, often in a garbled fashion, he did not immediately leap on the bandwagon. It was probably a relief to him when, in the 1990s, it became clear that at least some of the secrets were no longer official. At his eightieth birthday celebrations in 1997 he felt able to talk informally about some of the details, and in 1998 he was persuaded to give a talk at the opening of the Center for Applied Cryptographic Research at the University of Waterloo.

In this talk [10], entitled *Fish and I*, he tells how, others having failed, he was asked to work on the cipher system known at Bletchley as Tunny. This was one of the 'Fish' codes used by the German High Command. He had an idea and, although not optimistic, he 'thought it best to seem busy'. So he copied out some ciphertext onto sheets of squared paper, using chunks of various lengths, noticed certain patterns, and was able to infer the structure of the system. Indeed, he achieved a virtual reconstruction of an extremely complex machine using only scraps of information – an amazing feat that must rank as one of his greatest intellectual achievements.

The success of Bletchley as an institution was partly due to the fact that the powers-that-be were not stupid, and soon many people were helping to work out the implications of Tutte's discovery. This continued throughout 1942 and 1943, with regular upgrading of the techniques to deal with improvements in the system. Eventually it became necessary to use a form of number-crunching statistical analysis, and Tutte saw how this could be done. He reported his ideas and, in his own words, 'there were rapid developments'. The outcome was that the famous Colossus computer was deployed on these problems.

At the end of the War, Trinity College elected Tutte to a Research Fellowship in Mathematics. Although less prestigious in the public eye than the awards given to other civil servants, it was probably more appreciated by the recipient. Exactly how it came about is unclear. C.A.B. Smith recalled being stopped in the street by a Fellow of Trinity, who said 'we've just elected Tutte to a Fellowship but we don't know what he has done or where he lives' [14].

The period at Trinity was a highly productive one. His Ph.D. thesis on 'An Algebraic Theory of Graphs' contained many seminal ideas, and these were published in papers that quickly established graph theory as a significant area of mathematics, with Tutte as its master builder. Among the papers that were published at that time there are several classics. In a paper published in 1946 [2] he disproved Tait's conjecture by constructing a planar cubic graph that has no Hamilton cycle. His paper on the symmetry of cubic graphs [3] contains a truly unexpected bound on the order of a vertex-stabilizer, a fact that was to resurface twenty years later in the work of permutation-group-theorists. Perhaps the most influential paper is the one on factorization of graphs [4], in which he obtains the canonical form of the basic result on this topic, with Petersen's theorem as a simple corollary.

Much later, in his book *Graph Theory As I Have Known It* [11], he gave a fascinating account of how he arrived at some of these fundamental results. Perhaps not surprisingly, it was often by a process that offered an intellectual challenge rather than a guarantee of success. The use of Pfaffians in the proof of the factorization theorem was marvellous, even if it was later shown to be unnecessary. As well as graph theory, his thesis also contained important results about matroids, a subject that had been inaugurated by Hassler Whitney. Many of these results were published about ten years later [7], but their significance was not fully recognized until they appeared in a series of lectures in 1965 [8].

In 1948 he took up a post at the University of Toronto. Here he continued to produce a stream of new ideas and, rather unexpectedly, for he was a very shy man, he got married. His wife Dorothea would bemoan the fact that weekends had to be spent on research, because Bill feared that mathematical inspiration would dry up before he was 40 (at least, that's what he told her). Some of his Toronto papers discussed aspects of the chromatic polynomial and its two-variable generalization, now known (justifiably) as the Tutte polynomial [5]. Several famous conjectures, such as the conjecture that every bridgeless graph has a 5-flow, also appeared in print at this time.

In 1962 he was persuaded to move to the newly-established University of Waterloo. By this time he had been appointed a Fellow of the Royal Society of Canada, and his eminence was being recognised internationally. The university created around him a world-famous Department of Combinatorics and Optimisation, and it was instrumental in the foundation of the *Journal of Combinatorial Theory*. He himself was not an administrator, but he supported and encouraged people whose talents lay in that direction, and his placid temperament helped to calm the troubled waters that sometimes threatened his Department.

By the 1970s the growth of air travel meant that Bill and Dorothea were able to travel extensively, and they returned to England on several occasions. In 1971 he was the principal guest at a small meeting held in Royal Holloway College. The success of that meeting led to the establishment of the continuing series of British Combinatorial Conferences, and Bill spoke at the first one to be organised on a regular basis, the Fourth BCC in Aberystwyth (1973).

His work at this period centred on the enumeration of planar graphs, and specifically four-colourable planar graphs. There was a slight chance that the four-colour conjecture could be settled 'asymptotically', but he did not have great hopes for the method. He greeted the Appel-Haken resolution of the conjecture enthusiastically, agreeing that the strategy was sound, even if the calculations could not be checked by hand [9].

He retired formally in 1985, but continued to be active in mathematics. In his quiet way he enjoyed the recognition that accompanied the growth in popularity and status of Graph Theory, the subject he had built. Outstanding mathematicians were attracted to work in this field, many of them inspired by Tutte's earlier results.

After Dorothea's death in 1994 he lived in England for a while, but he did not settle, and eventually returned to his adopted home in Ontario. It was proper that his eightieth birthday should be marked by a celebration in Waterloo where he was able to talk about his work to an audience that fully appreciated what he had achieved. In Britain we were fortunate to have him as the Rado Lecturer at the BCC in Canterbury in 1999, where he spoke about *The Coming of the Matroids* [12]. In this talk he explained how some of his work at Bletchley had helped him to understand the properties of linear dependence, and how this led to some of the fundamental theorems of matroid theory.

In 2001 his eminence was recognised by the award of the Order of Canada, which he received with characteristic humour and humility. At that time he was in good health, but in March 2002 he was diagnosed with a serious medical condition, and he died on May 2, in his 85th year.

Author's Note This is an extended version of the obituary published in *The Independent* on 9 May 2002. I am grateful to several people, in particular Dan Younger, for additional information. A full appreciation of Tutte's mathematical work is planned to appear in the *Bulletin of the London Mathematical Society*.

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Decompositions of complete graphs: embedding partial edge-colourings and the method of amalgamations

L.D. Andersen and C.A. Rodger

Abstract

We consider decompositions of the edges of complete graphs, mainly with each part inducing a spanning subgraph with specified properties. We give special attention to embedding questions, where a decomposition of the edges of a complete subgraph of a complete graph is given, and the problem is to extend the decomposition to a full decomposition with the desired properties. One way to obtain decomposition and embedding results simultaneously is by the method of amalgamations of vertices, to which we devote a large part of the paper.

1 Introduction

1.1 Some sample questions

As a starting point, consider the edge-coloured complete graph K_6 in Figure 1. Its edge-colouring (indicated by numbers) is proper, meaning that each colour occurs on at most one edge incident with each vertex, so thinking of the graph as a subgraph of a K_{11} it makes sense to ask

1. Can the edge-colouring be completed to a proper edge-colouring of K_{11} with the least possible number of colours, 11?

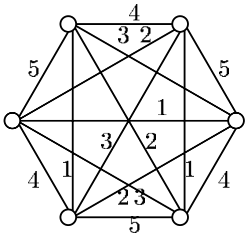

Figure 1

On the other hand, in K_{11} the colouring could easily be part of a colouring of the edges which is not necessarily proper, but which may have other nice properties, such as each colour class inducing a given subgraph or a subgraph with some given property. In this spirit, one could ask

2. Can the edge-colouring be completed to an edge-colouring of K_{11} in which every colour class induces a spanning connected subgraph?

3. Can the edge-colouring be completed to an edge-colouring of K_{11} in which every colour class induces a Hamiltonian cycle?

4. Can the edge-colouring be completed to an edge-colouring of K_{11} in which every colour class induces the disjoint union of a 6-cycle and a 5-cycle?

1.2 Embeddings

Many questions can be asked about completing partial structures such as edge-colourings. In this paper, we focus on situations where the prescribed part is concentrated in a given part of the larger structure, and consequently we shall mainly talk about *embeddings* rather than completions (not that this distinction matters much).

Proper edge-colourings are models of timetables for pairwise meetings and, in the case of complete graphs, round-robin tournaments. In such connections, it is natural to wish to extend smaller schedules to larger ones, giving rise to questions like sample question 1 above. In many cases the derived problems, for example *'When can a proper edge-colouring of K_r be embedded in a proper edge-colouring of K_n with the minimum number of colours?'* are not hard, but they can be extremely challenging with small additional conditions such as requiring a given set of mutually independent edges in $K_n \setminus K_r$ to have prescribed colours. This kind of condition arises naturally when the embedding result is developed as a tool for solving other combinatorial problems.

Quite generally, considering embedding problems for combinatorial structures is both a natural and a fruitful exercise. Thus embedding problems have been considered for a variety of such structures: partial quasigroups (latin squares), edge-coloured graphs, partial balanced incomplete block designs, and many more. The insight gained can have many facets: can smaller structures be simple building blocks in larger structures, does the embedding method give an algorithm for construction of big structures — or maybe even an existence proof? There are two basic questions that should perhaps be asked first when considering embedding problems for a given type of combinatorial structure.

The first basic question is: *'Given a small structure of the type considered, is there always a finite embedding?'* (We keep everything finite in this paper.) Again, a question such as *'Does there exist an n such that any proper edge-colouring of K_r can be embedded in a proper edge-colouring of K_n with the minimum number of colours?'* is not too hard, but other more difficult, yet analogous questions can be asked.

We usually require the target structure to be complete in some sense, such as a latin square (not partial), a proper edge-colouring with the minimum number of colours, and so on. The second natural question applies when the smaller structure is complete as well; for example *'What is the smallest larger latin square into which a latin square can be embedded?'* Here we have a simple, clean version of the general problem, where one would expect the answer to be more easily found than in the general case. In sample question 1, we actually have this situation, as the graph of Figure 1 is properly coloured with the minimum number of colours, 5. And it is easy to see that it cannot be embedded in the desired way in a complete graph with fewer than 11 vertices.

The embedding problems related to sample question 1 have a tight relationship to the theory of latin squares, as this is where the most obvious previous analogues are to be found, and also where some of the questions demanding more subtle embedding results arose. We survey some of the latin square analogues in Section 2, and then devote Section 3 to the treatment of sample question 1 and extensions of it.

1.3 Amalgamations

For the remaining three sample questions, or rather for the vast class of problems for which they are just simple examples, we describe the *method of amalgamations*. Designed to address more general questions, this method — when it works — almost always produces an embedding result as well. It was first introduced in connection with latin squares, but in 1984 Hilton [27] showed how to construct Hamiltonian decompositions of complete graphs with it.

When used on graphs, the point of the method is to find a clever description of what the target edge-colouring (such as a Hamiltonian decomposition with a colour for each Hamiltonian cycle) looks like when subsets of vertices are each identified, that is *amalgamated* to form a single vertex. It is then often easier to recognise such an amalgamated colouring in a smaller graph, and the strength of the method is in situations where it can be shown that, given what might be an amalgamated colouring, the vertices can indeed be pulled apart again to give the desired result. We elaborate on this in Section 4. It is important to note that the method involves both multiple edges and loops, so *multiple edges and loops are allowed in the graphs of this paper.*

The embedding corollaries are obtained from the method of amalgamations when all vertices outside the graph to be embedded are amalgamated into one single vertex.

Since Hilton's paper, itself covering sample question 3, the method has been extended and used to address questions such as sample question 2. Sample question 4 is an example of an embedding problem derived from the *Oberwolfach problem.* This problem asks whether the complete graph K_{2n+1} can be decomposed into isomorphic spanning subgraphs, each being the union of disjoint cycles of specified lengths. (The cycle lengths correspond to hypothetical table sizes, in terms of numbers of persons seated at the table, at the mathematical research institute in Oberwolfach in Germany; the vertices of the complete graph correspond to the participants in a given meeting; and the decomposition is a seating schedule ensuring that, over a number of meals, each participant sits next to each other participant exactly once.) The Oberwolfach problem is not solved, and the ultimate goal of using the method of amalgamations on it would be to obtain a solution. So here it is the existence problem that is in focus, not the embedding.

1.4 Flashback

Still, we begin by looking at embedding results, old and new, in the style of sample question 1. The inspiration comes very much from latin squares, so we take a look at these first.

By the way, the answer to all four sample questions in Subsection 1.1 is yes . . . .

2 The heritage from latin squares

2.1 Latin square terminology

A *latin square* of side n is an $n \times n$ matrix with entries from an n-set, in which each symbol occurs exactly once in each row and exactly once in each column. We shall define two partial structures: a *partial latin square* of side n is an $n \times n$ matrix in which some cells may be empty and the rest have entries from an n-set in such a way that each symbol occurs at most once in each row and at most once in each column, and a *latin rectangle* of size $r \times s$ on n symbols is an $r \times s$ matrix with entries from an n-set, where each symbol occurs at most once in each row and at most once in each column.

Thus latin rectangles have no empty cells. The theorems mentioned in the following are mainly concerned with latin rectangles, sometimes with a few elements outside a latin rectangle also prescribed.

We note that a latin square of side n is equivalent to a proper edge-colouring of the complete bipartite graph $K_{n,n}$ with n colours (one such equivalence is obtained by taking rows as one bipartition class of vertices, columns as the other, and symbols as colours, but in fact the roles of rows, columns and symbols can be permuted).

A latin square, a partial latin square or a latin rectangle is said to be *symmetric* if cell (i, j) always contains the same entry as cell (j, i).

A symmetric latin square of side n corresponds to a proper edge-colouring with n colours of the complete graph K_n with a loop on each vertex, the loops corresponding to the diagonal cells.

Remark In a symmetric latin square of even side, each symbol occurs an even number of times on the diagonal. In a symmetric latin square of odd side, each symbol occurs exactly once on the diagonal.

2.2 The first embedding theorem

Although this whole excursion into latin squares is meant to be brief, we dwell a little on the following theorem, proved by Hall [24] in 1945.

Theorem 2.1 *Every latin rectangle of size $r \times n$ on n symbols can be embedded in a latin square of side n.*

A consequence of this simple theorem is a very simple way of constructing latin squares: it can be done row by row. Write down any permutation of the n symbols, and you have a latin rectangle of size $1 \times n$, and by the theorem it can be completed to a latin square of side n. Find, in any way, a second permutation, with no repetition in any column when used as the second row, and so on. Thus the theorem is also an existence proof for latin squares! (Although the existence problem never was very hard . . . .)

The result can be interpreted as stating that a complete bipartite graph $K_{n,n}$ can be properly edge-coloured by repeatedly giving a 1-factor a new colour and then deleting it, and so it is generalised by Kőnig's theorem that a bipartite graph with maximum degree Δ has a proper edge-colouring with Δ colours. This theorem also gives an easy proof of Theorem 2.1 (in Section 4.2, we present a generalisation of Kőnig's theorem needed for the method of amalgamations). Another proof of Hall's theorem, echoing more the row-by-row completion, is obtained by always constructing a new row by finding a system of distinct representatives for the sets of symbols not yet appearing in each column.

Both these techniques can be used to prove the latin square embedding results of the next subsection as well.

2.3 Ryser type embeddings

If R is a latin rectangle, we let $R(\sigma)$ denote the number of cells of R occupied by the symbol σ.

Ryser proved the following generalisation of Theorem 2.1 in 1951.

Theorem 2.2 ([55]) *A latin rectangle R of size $r \times s$ on n symbols can be embedded in a latin square of side n if and only if*

$$R(\sigma) \geqslant r + s - n$$

for all symbols σ.

In an influential paper [21] from 1960, Evans formulated the obvious corollary.

Corollary 2.3 *A latin rectangle of size $r \times r$ on n symbols can be embedded in a latin square of side n for all $n \geqslant 2r$, and this bound is best possible.*

The graph theoretic translation of these results deals with complete bipartite graphs, and it is not our intention to pursue the connection in this paper. However, in 1974, Cruse [17] proved analogues of both for symmetric latin squares.