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Stability in Hamiltonian Systems:

Applications to the restricted three-body problem

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Based on lectures by Ken Meyer

1 Introduction

As participants in the MASIE-project, we attended the summer school *Mechanics and Symmetry* in Peyresq, France, during the first two weeks of September 2000. These lecture notes are based on the notes we took there from Professor Meyer's lecture series *"N-Body Problems"*.

The N-body problem is a famous classical problem. It consists in describing the motion of N planets that interact with a gravitational force. Already in 1772, Euler described the three-body problem in his effort to study the motion of the moon. In 1836 Jacobi brought forward an even more specific part of the three body problem, namely that in which one of the planets has a very small mass. This system is the topic of this paper and is nowadays called the *restricted three-body problem*. It is a conservative system with two degrees of freedom, which gained extensive study in mechanics.

The N-body problem has always been a major topic in mathematics and physics. In 1858, Dirichlet claimed to have found a general method to treat any problem in mechanics. In particular, he said to have proven the stability of the planetary system. This statement is still questionable because he passed away without leaving any proof. Nevertheless, it initiated Weierstrass and his students Kovalevski and Mittag-Leffler to try and rediscover the method mentioned by Dirichlet. Mittag-Leffler even managed to convince the King of Sweden and Norway to establish a prize for finding a series expansion for coordinates of the N-body problem valid for all time, as indicated by Dirichlet's statement. In 1889, this prize was awarded to Poincaré, although he did not solve the problem. His essay, however, produced a lot of original ideas which later turned out to be very important for mechanics. Moreover, some of them even stimulated other branches of mathematics, for instance topology, to be born and later on gain extensive study. Despite of all this effort, the N-body problem is still unsolved for N>2[1].

This paper focuses on the relatively simple restricted three-body problem. This describes the motion of a test particle in the combined gravitational field of two planets and it could serve for instance as a model for the motion of a satellite in the Earth-Moon system or a comet in the Sun-Jupiter system. The restricted three-body problem has a number of relative equilibria, which we compute. The remaining text will mainly be concerned with general Hamiltonian equilibria. Stability criteria for these equilibria will be derived, as well as detection methods for bifurcations of periodic solutions. Classical and more advanced mathematical techniques are used, such as spectral analysis, Liapunov functions, Birkhoff-Gustavson normal forms, Poincaré sections, and Kolmogorov twist stability. All help to study the motion of the test particle near the relative equilibria of the restricted problem.

2 The restricted three-body problem

Before introducing the restricted three-body problem, let us study the two-body problem, the motion of two planets interacting via gravitation. Denote by $\boldsymbol{X_1}, \boldsymbol{X_2} \in \mathbb{R}^3$ the positions of the planets 1 and 2 respectively. Let us assume that planet 1 has mass $0 < \mu < 1$, planet 2 has mass $1 - \mu$ and the gravitational constant is equal to 1. These assumptions are not very restrictive, because they can always be arranged by a rescaling of time. The equations of

[1] Summarized from [10], [11] and [8]

motion for the two-body problem then read:

$$\begin{aligned} \frac{d^2\boldsymbol{X_1}}{dt^2} &= -\tfrac{(1-\mu)}{||\boldsymbol{X_1}-\boldsymbol{X_2}||^3}(\boldsymbol{X_1}-\boldsymbol{X_2}) \\ \frac{d^2\boldsymbol{X_2}}{dt^2} &= -\tfrac{\mu}{||\boldsymbol{X_1}-\boldsymbol{X_2}||^3}(\boldsymbol{X_2}-\boldsymbol{X_1})\,. \end{aligned} \tag{2.1}$$

Let us denote the center of mass

$$\boldsymbol{Z} := \mu\boldsymbol{X_1} + (1-\mu)\boldsymbol{X_2}\,. \tag{2.2}$$

Then we derive from (2.1) and (2.2) that $\frac{d^2\boldsymbol{Z}}{dt^2} = 0$, expressing that the center of mass moves with constant speed. Now we transform to co-moving coordinates

$$\boldsymbol{Y_i} = \boldsymbol{X_i} - \boldsymbol{Z} \text{ for } i = 1, 2, \tag{2.3}$$

and we write down the equations of motions in these new variables:

$$\frac{d^2\boldsymbol{Y_1}}{dt^2} = -\frac{(1-\mu)^3}{||\boldsymbol{Y_1}||^3}\boldsymbol{Y_1}\,, \qquad \frac{d^2\boldsymbol{Y_2}}{dt^2} = -\frac{\mu^3}{||\boldsymbol{Y_2}||^3}\boldsymbol{Y_2}\,. \tag{2.4}$$

Let us analyze these equations a bit more. First of all, we see from the definitions (2.2) and (2.3) that $\mu\boldsymbol{Y_1} + (1-\mu)\boldsymbol{Y_2} = 0$, so $\boldsymbol{Y_1}$ and $\boldsymbol{Y_2}$ lie on a line through the origin of $\mathbb{R}^3$, both at another side of the origin, and their length ratio $\frac{||\boldsymbol{Y_1}||}{||\boldsymbol{Y_2}||}$ is fixed to the value $\frac{1-\mu}{\mu}$. The line connecting $\boldsymbol{Y_1}, \boldsymbol{Y_2}$ and the origin is called the *line of syzygy*. Because $\boldsymbol{Y}_2 = -\frac{\mu}{1-\mu}\boldsymbol{Y}_1$, we in fact only need to study the first equation of (2.4). The motion of the second planet then follows automatically.

Secondly, by differentiation one finds that the angular momentum $\boldsymbol{Y_1} \times \frac{d\boldsymbol{Y_1}}{dt}$ is independent of time. Indeed, $\frac{d}{dt}(\boldsymbol{Y_1} \times \frac{d\boldsymbol{Y_1}}{dt}) = \frac{d\boldsymbol{Y_1}}{dt} \times \frac{d\boldsymbol{Y_1}}{dt} + \boldsymbol{Y_1} \times \frac{d^2\boldsymbol{Y_1}}{dt^2} = 0$, because both terms are the cross-products of collinear vectors.

In the case that $\boldsymbol{Y_1} \times \frac{d\boldsymbol{Y_1}}{dt} = 0$, and assuming that $\boldsymbol{Y_1}(0) \neq 0$, we have that $\frac{d\boldsymbol{Y_1}}{dt}$ has the same direction as $\boldsymbol{Y_1}$, so the motion takes place in a one-dimensional subspace: $\boldsymbol{Y_1}, \frac{d\boldsymbol{Y_1}}{dt}, \boldsymbol{Y_2}, \frac{d\boldsymbol{Y_2}}{dt} \in \boldsymbol{Y_1}(0)\mathbb{R} = \boldsymbol{Y_2}(0)\mathbb{R}$. It is not difficult to derive the following scalar second order differential equation for the motion in this subspace: $\frac{d^2}{dt^2}||\boldsymbol{Y_1}|| = -(1-\mu)^3/||\boldsymbol{Y_1}||^2$. It turns out that in this case $\boldsymbol{Y_1}$ and $\boldsymbol{Y_2}$ fall into the origin in a finite time.

In the case that $\boldsymbol{Y_1} \times \frac{d\boldsymbol{Y_1}}{dt} \neq 0$, the motion takes place in the plane perpendicular to $\boldsymbol{Y_1} \times \frac{d\boldsymbol{Y_1}}{dt}$, because both $\boldsymbol{Y_1}$ and $\frac{d\boldsymbol{Y_1}}{dt}$ are perpendicular to the constant vector $\boldsymbol{Y_1} \times \frac{d\boldsymbol{Y_1}}{dt}$. By rotating our coordinate frame, we can arrange that $\boldsymbol{Y_1} \times \frac{d\boldsymbol{Y_1}}{dt}$ is some multiple of the third basis vector. Thus we can consider the equations (2.4) as two second order planar equations. It is well-known that the planar solutions of $\frac{d^2\boldsymbol{Y_1}}{dt^2} = -\frac{(1-\mu)^3}{||\boldsymbol{Y_1}||^3}\boldsymbol{Y_1}$ with $\boldsymbol{Y_1} \times \frac{d\boldsymbol{Y_1}}{dt} \neq 0$ describe one of the conic sections: a circle, an ellipse, a parabola or a hyperbola. $\boldsymbol{Y_2}$ clearly describes a similar conic section.

Let us now assume that a certain solution of the two-body problem is given to us. We want to study the motion of a *test particle* in the gravitational field of the two main bodies, which we call *primaries*. The test particle is assumed to have zero mass. Therefore it does not affect the primaries, but it does feel the gravitational force of the primaries acting on it. The resulting problem is called the restricted three-body problem. It could serve as a model for a satellite in the Earth-Moon system or a comet in the Sun-Jupiter system. Let $\boldsymbol{X} \in \mathbb{R}^3$ denote the position of the test particle. Then the restricted three-body problem is given by

$$\frac{d^2\boldsymbol{X}}{dt^2} = -\frac{\mu}{||\boldsymbol{X}-\boldsymbol{X_1}||^3}(\boldsymbol{X}-\boldsymbol{X_1}) - \frac{(1-\mu)}{||\boldsymbol{X}-\boldsymbol{X_2}||^3}(\boldsymbol{X}-\boldsymbol{X_2}) , \tag{2.5}$$

in which $(\boldsymbol{X_1}, \boldsymbol{X_2})$ is the given solution of the two-body problem. One can again transform to co-moving coordinates, setting $\boldsymbol{Y} = \boldsymbol{X} - \boldsymbol{Z}$, which results in the system

$$\frac{d^2\boldsymbol{Y}}{dt^2} = -\frac{\mu}{||\boldsymbol{Y}-\boldsymbol{Y_1}||^3}(\boldsymbol{Y}-\boldsymbol{Y_1}) - \frac{(1-\mu)}{||\boldsymbol{Y}-\boldsymbol{Y_2}||^3}(\boldsymbol{Y}-\boldsymbol{Y_2}) . \tag{2.6}$$

At this point we start making assumptions. Let us assume that the primaries move in a circular orbit around their center of mass with constant angular velocity . This is approximately true for the Earth-Moon system and the Sun-Jupiter system. We set the angular velocity equal to 1. Without loss of generality, we can assume that the motion of the primaries takes place in the plane perpendicular to the third basis-vector. Thus, after translating time if necessary,

$$\boldsymbol{Y_1} = R(t)\begin{pmatrix} 1-\mu \\ 0 \\ 0 \end{pmatrix} , \quad \boldsymbol{Y_2} = R(t)\begin{pmatrix} -\mu \\ 0 \\ 0 \end{pmatrix} , \tag{2.7}$$

in which $R(t)$ is the rotation matrix:

$$R(t) := \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{2.8}$$

Note that we have introduced a rotating coordinate frame in which the motion of the primaries has become stationary. At this point we put in our test particle and again we make an assumption, namely that it moves in the same plane as the primaries do. So we set

$$\boldsymbol{Y} = R(t) \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} . \tag{2.9}$$

Let $(\boldsymbol{x}, 0)^T = (x_1, x_2, 0)^T$ be the coordinates of the test particle in the rotating coordinate frame. By inserting (2.7), (2.8) and (2.9) into (2.6), multiplying the resulting equation from the left by $R(t)^{-1}$ and using two following identities

$$\frac{d^2}{dt^2} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} = - \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} ,$$

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}^{-1} \frac{d}{dt} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} ,$$

we deduce the planar equations of motion for $\boldsymbol{x} \in \mathbb{R}^2$:

$$\frac{d^2\boldsymbol{x}}{dt^2} - \boldsymbol{x} + \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \frac{d\boldsymbol{x}}{dt} = \\ -\frac{\mu}{||\boldsymbol{x} - \binom{1-\mu}{0}||^3} \left(\boldsymbol{x} - \binom{1-\mu}{0}\right) - \frac{1-\mu}{||\boldsymbol{x} - \binom{-\mu}{0}||^3} (\boldsymbol{x} - \binom{-\mu}{0}).$$

Finally, setting $\boldsymbol{y} = \frac{d\boldsymbol{x}}{dt} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \boldsymbol{x}$, we find that these are Hamiltonian equations of motion on $\mathbb{R}^4 \backslash \{\boldsymbol{x} = \binom{1-\mu}{0}, \binom{-\mu}{0}\}$ with Hamiltonian

$$H = \frac{1}{2}(y_1^2 + y_2^2) - (x_1 y_2 - x_2 y_1) - \frac{\mu}{||\boldsymbol{x} - \binom{1-\mu}{0}||} - \frac{1-\mu}{||\boldsymbol{x} - \binom{-\mu}{0}||} , \tag{2.10}$$

where we have equipped $\mathbb{R}^4$ with the canonical symplectic form $dx_1 \wedge dy_1 + dx_2 \wedge dy_2$, i.e. the equations of motion are given by $\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}$, $\frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}$.

3 Relative equilibria

Let us look for equilibrium solutions of the Hamiltonian vector field induced by (2.10). These correspond to stationary motion of the test particle relative to the rotating coordinate frame and are therefore called *relative equilibria*. In the original coordinates they correspond to the test particle rotating around the center of mass of the primaries with angular velocity 1.

First of all, to facilitate notation, we introduce the potential energy function

$$V(\boldsymbol{x}) := -\frac{\mu}{||\boldsymbol{x} - \binom{1-\mu}{0}||} - \frac{1-\mu}{||\boldsymbol{x} - \binom{-\mu}{0}||}.$$

To find the equilibrium solutions of (2.10) we set all the partial derivatives of H equal to zero and find

$$y_1 + x_2 = 0\ ,\ y_2 - x_1 = 0\ ,\ -y_2 + \frac{\partial V}{\partial x_1}(\boldsymbol{x}) = 0\ ,\ y_1 + \frac{\partial V}{\partial x_2}(\boldsymbol{x}) = 0\ ,$$

or equivalently,

$$\frac{\partial V}{\partial x_1}(\boldsymbol{x}) = x_1\ ,\ \frac{\partial V}{\partial x_2}(\boldsymbol{x}) = x_2\ , \tag{3.1}$$

where $\boldsymbol{y}$ at the equilibrium point can easily be found once we solved (3.1) for $\boldsymbol{x}$ at the equilibrium point. Note that $\boldsymbol{x}$ solves (3.1) if and only if $\boldsymbol{x}$ is a stationary point of the function

$$U(\boldsymbol{x}) := \frac{1}{2}(x_1^2 + x_2^2) - V(\boldsymbol{x})\ ,$$

called the *amended potential*.

Let us first look for equilibrium points of the amended potential that lie on the line of syzygy, i.e. for which $x_2 = 0$. Note that $\frac{\partial U}{\partial x_2}(\boldsymbol{x}) = 0$ is automatically satisfied in this case since $\frac{\partial V}{\partial x_2}|_{x_2=0} \equiv 0$. $\frac{\partial U}{\partial x_1}(\boldsymbol{x}) = 0$ reduces to

$$\frac{d}{dx_1}U(x_1,0) = \frac{d}{dx_1}\left(\frac{1}{2}x_1^2 + \frac{\mu}{|x_1+\mu-1|} + \frac{1-\mu}{|x_1+\mu|}\right) = 0\ . \tag{3.2}$$

Clearly, $U(x_1,0)$ goes to infinity if x_1 approaches $-\infty, -\mu, 1-\mu$ or ∞, so $U(x_1,0)$ has at least one critical point on each of the intervals $(-\infty,-\mu)$, $(-\mu, 1-\mu)$ and $(1-\mu,\infty)$. But we also calculate that $\frac{d^2}{dx_1^2}U(x_1,0) = 1 + 2\frac{\mu}{|x_1+\mu-1|^3} + 2\frac{1-\mu}{|x_1+\mu|^3} > 0$. So $U(x_1,0)$ is convex on each of these intervals and we conclude that there is exactly one critical point in each of the intervals. The three relative equilibria on the line of syzygy are called the *Eulerian* equilibria. They are denoted by $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$, where $\mathcal{L}_1 \in (-\infty,-\mu)\times\{0\}$, $\mathcal{L}_2 \in (-\mu,1-\mu)\times\{0\}$ and $\mathcal{L}_3 \in (1-\mu,\infty)\times\{0\}$.

Now we shall look for equilibrium points that do not lie on the line of syzygy. Let us use $d_1 = ||\boldsymbol{x} - \binom{1-\mu}{0}|| = \sqrt{(x_1+\mu-1)^2 + x_2^2}$ and $d_2 = ||\boldsymbol{x} - \binom{-\mu}{0}|| = \sqrt{(x_1+\mu)^2 + x_2^2}$ as coordinates in each of the half-planes $\{x_2 > 0\}$ and $\{x_2 < 0\}$. Then U can be written as $U = \frac{\mu}{2}d_1^2 + \frac{1-\mu}{2}d_2^2 - \frac{\mu(1-\mu)}{2} + \frac{\mu}{d_1} + \frac{1-\mu}{d_2}$. So the critical points of U are given by $d_i = d_i^{-2}$ i.e. $d_1 = d_2 = 1$. This gives us the two *Lagrangean* equilibria which lie at the third vertex of the equilateral triangle with the primaries at its base-points: $\mathcal{L}_4 = (\frac{1}{2}-\mu, \frac{1}{2}\sqrt{3})^T$ and $\mathcal{L}_5 = (\frac{1}{2}-\mu, -\frac{1}{2}\sqrt{3})^T$.

This paper discusses some useful tools for the study of the flow of Hamiltonian vector fields near equilibrium points. We will for instance establish stability criteria for Hamiltonian equilibria and study bifurcations of periodic solutions near Hamiltonian equilibria. The Eulerian and Lagrangean equilibria of the restricted three-body problem will serve as an instructive and inspiring example.

4 Linear Hamiltonian systems

One of the techniques to prove stability for an equilibrium of a system of differential equations, is to analyze the linearized system around that equilibrium. Stability or instability then may follow from the eigenvalues of the matrix of the linearized system. In Hamiltonian systems, these eigenvalues have a special structure which implies that the linear theory can only be used to prove instability, not stability. We will start by giving a brief introduction to linear Hamiltonian systems. We then conclude this section with a lemma which shows why one can not conclude stability from the linear analysis.

Consider a symplectic vector space $\mathbb{R}^{2n}$ with coordinates $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y})^T$ and the symplectic form is $d\boldsymbol{x} \wedge d\boldsymbol{y} := \sum_{j=1}^n dx_j \wedge dy_j$. Then every continuously differentiable function $H : \mathbb{R}^{2n} \to \mathbb{R}$ induces the Hamiltonian vector field X_H on $\mathbb{R}^{2n}$ defined by $X_H(z) = J(\nabla H(z))^T$, in which the $2n \times 2n$ matrix

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

is called the *standard symplectic matrix*. Note that X_H gives rise to the Hamiltonian system of differential equations $\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}$, $\frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}$. The function H is called the Hamiltonian function of the vector field X_H.

Suppose that for $\boldsymbol{z}_\circ \in \mathbb{R}^{2n}$ we have $\nabla H(\boldsymbol{z}_\circ) = 0$, then $\boldsymbol{z}_\circ$ is called a *rest point*, *equilibrium point*, *fixed point*, or *critical point* of H. Note that $X_H(\boldsymbol{z}_\circ) = 0$ so $\boldsymbol{z}_\circ$ is fixed by the flow of X_H. By translating our coordinate frame, we can arrange that $\boldsymbol{z}_\circ = 0$. We will assume that H is a sufficiently smooth function in a neighborhood of its equilibrium 0, so that we can write $H(\boldsymbol{z}) = H_2(\boldsymbol{z}) + \mathcal{O}(||\boldsymbol{z}||^3)$ as $\boldsymbol{z} \to 0$, where H_2 is a quadratic form on $\mathbb{R}^{2n}$. The linearized vector field of X_H at 0 is the Hamiltonian vector field X_{H_2} induced by the quadratic Hamiltonian H_2. This encourages us to study quadratic Hamiltonians and their induced linear Hamiltonian vector fields.

Let $H_2 : \mathbb{R}^{2n} \to \mathbb{R}$ be a quadratic form, determined by the symmetric $2n \times 2n$ matrix Q, i.e. $H_2(\boldsymbol{z}) = \frac{1}{2}\boldsymbol{z}^T Q \boldsymbol{z}$ with $Q^T = Q$. H_2 generates a linear Hamiltonian vector field:

$$X_{H_2}(\boldsymbol{z}) = J(\nabla H_2(\boldsymbol{z}))^T = JQ\boldsymbol{z}\,. \tag{4.1}$$

Matrices S of the form $S = JQ$ for some symmetric matrix Q are called *infinitesimally symplectic* or *Hamiltonian*. The set of all infinitesimally symplectic matrices is denoted by

$$\begin{aligned}\mathfrak{sp}(n) &:= \{S \in \mathbb{R}^{2n\times 2n} \mid S = JQ \text{ for some } Q = Q^T\} \\ &= \{S \in \mathbb{R}^{2n\times 2n} \mid S^T J + JS = 0\}\ .\end{aligned}$$

Note that the standard symplectic matrix J satisfies $J^{-1} = J^T = -J$. Now take any infinitesimally symplectic matrix S of the form $S = JQ$, with Q symmetric. Then the simple calculation

$$J^{-1}(-S^T)J = J^{-1}(-JQ)^T J = -J^{-1}(QJ^T)J = -J^{-1}Q = JQ = S,$$

shows that S and $-S^T$ are similar. But similar matrices have equal eigenvalues. And because S has real coefficients, this observation leads to the following lemma:

Lemma 4.1 *If $S \in \mathfrak{sp}(n)$ and λ is an eigenvalue of S, then also $-\lambda, \overline{\lambda}$ and $-\overline{\lambda}$ are eigenvalues of S.*

Now let us consider the exponential of an infinitesimally symplectic matrix, $\exp(S) = \exp(JQ)$, which is the fundamental matrix for the time-1 flow of the linear Hamiltonian vector field $\boldsymbol{z} \mapsto S\boldsymbol{z} = JQ\boldsymbol{z}$. It is a nice exercise to show that it satisfies $(\exp(S))^T J \exp(S) = J$. In general, a matrix $P \in \mathbb{R}^{2n\times 2n}$ satisfying $P^T JP = J$ is called *symplectic*. The set of symplectic matrices is denoted

$$\mathrm{Sp}(n) := \{P \in \mathbb{R}^{2n\times 2n} \mid P^T JP = J\}.$$

For a symplectic matrix P one easily derives that $J^{-1}P^{-T}J = P$, so P^{-T} and P are similar. This leads to:

Lemma 4.2 *If $P \in \mathrm{Sp}(n)$ and λ is an eigenvalue of P, then so too are $\lambda^{-1}, \overline{\lambda}$ and $\overline{\lambda}^{-1}$.*

We remark here that $\mathrm{Sp}(n)$ is a Lie group with matrix multiplication. Its Lie algebra is exactly $\mathfrak{sp}(n)$.

Remember that we studied linear Hamiltonian systems to determine stability or instability of an equilibrium from the spectrum of its linearized vector field. From lemma 4.1 we see if one eigenvalue has a nonzero real part, then there must be an eigenvalue with positive real part. In this case the equilibrium is unstable. The other possibility is that all eigenvalues are purely imaginary. In this case, adding nonlinear terms could destabilize the equilibrium. So lemma 4.1 states that for Hamiltonian systems, the linear theory can only be useful to prove instability of an equilibrium.

Lemma 4.2 states a similar thing for symplectic maps: if $\Psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a symplectic diffeomorphism with a fixed point, then the linearization of Ψ at that fixed point can only be used to prove instability of the fixed point, not stability.

The reader should be convinced now that we need more sophisticated mathematical techniques if we want to have stability results. Some of them will be explained in the following section.

5 Liapunov's and Chetaev's theorems

We will now describe a direct method to determine stability of an equilibrium. We will give references for the proofs and explain the interpretation of the theory instead. In section 6 we shall apply the obtained results to the relative equilibria of the restricted three-body problem.

Consider a general system of differential equations,

$$\dot{\boldsymbol{v}} = f(\boldsymbol{v}) \,, \tag{5.1}$$

where f is a C^r vector field on $\mathbb{R}^m$ and $f(0) = 0$. Let $V : U \to \mathbb{R}$ be a positive definite C^1 function on a neighborhood U of the origin, i.e. $V(0) = 0$ and $V(\boldsymbol{z}) > 0, \ \forall \boldsymbol{z} \in U \backslash \{0\}$. If $\boldsymbol{u}$ is a solution of (5.1), then the derivative of V along $\boldsymbol{u}$ is $\frac{d}{dt}V(\boldsymbol{u}(t)) = \nabla V(\boldsymbol{u}(t)) \cdot \dot{\boldsymbol{u}}(t) = \nabla V(\boldsymbol{u}(t)) \cdot f(\boldsymbol{u}(t))$. So let us define the *orbital derivative* $\dot{V} : U \to \mathbb{R}$ of V as

$$\dot{V}(\boldsymbol{v}) := \nabla V(\boldsymbol{v}) \cdot f(\boldsymbol{v}) \,.$$

Theorem 5.1 (Liapunov's theorem) *Given such a function V for the system of equations (5.1), we have:*

(i) *If $\dot{V}(\boldsymbol{v}) \leq 0, \ \forall \boldsymbol{v} \in U\backslash\{0\}$ then the origin is stable.*
(ii) *If $\dot{V}(\boldsymbol{v}) < 0, \ \forall \boldsymbol{v} \in U\backslash\{0\}$ then the origin is asymptotically stable.*
(iii) *If $\dot{V}(\boldsymbol{v}) > 0, \ \forall \boldsymbol{v} \in U\backslash\{0\}$ then the origin is unstable.*

The function V is called a *Liapunov function*.

Let us see what this means for $m = 2$. Since V is a positive definite function, 0 is a local minimum of V. This implies that there exists a small neighborhood U' of 0 such that the level sets of V lying in U' are closed curves. Recall that $\nabla V(\boldsymbol{u}_c)$ is a normal vector to the level set C of V at $\boldsymbol{u}_c$ pointing outward. If an orbit $\boldsymbol{u}(t)$ crosses this level curve C at $\boldsymbol{u}_c$, then the velocity vector of the orbit and the gradient $\nabla V(\boldsymbol{u}_c)$ will form an angle θ for which

$$\cos(\theta) = \frac{\dot{V}(\boldsymbol{u}_c)}{||\nabla V(\boldsymbol{u}_c)|| \, ||f(\boldsymbol{u}_c)||} \,.$$

$\dot{V}(\boldsymbol{u}) < 0$ implies that $\pi/2 < \theta < 3\pi/2$. It follows that the orbit is moving inwards the level curve C in this case. If $\dot{V}(\boldsymbol{u}) = 0$, the orbit follows C. If $\dot{V}(\boldsymbol{u}) > 0$ we see the orbit moving outwards of C, that is away from the origin. See [7] for proof of Liapunov's theorem.

An immediate implication of Liapunov's theorem is the following. Consider a Hamiltonian system

$$\dot{\boldsymbol{z}} = J(\nabla H(\boldsymbol{z}))^T \, . \tag{5.2}$$

A good candidate for the Liapunov function in this Hamiltonian system would be the Hamiltonian function itself, because the orbits of a Hamiltonian system lie in the level set of the Hamiltonian. So $\dot{V} = \dot{H} = 0$. Thus, if H is locally positive definite then Liapunov's theorem applies. And if H is negative definite, one can choose $-H$ as a Liapunov function. We have:

Theorem 5.2 (Dirichlet's Theorem) *The origin is a stable equilibrium of (5.2), if it is an isolated local maximum or local minimum of the Hamiltonian H.*

The condition for instability in Liapunov's theorem is very strong since it requires the orbital derivative to be positive everywhere in U. The following theorem is a way to conclude instability under somewhat weaker conditions.

Theorem 5.3 (Chetaev's theorem) *Let U be a small neighborhood of the origin where the C^1 Chetaev function $V : U \to \mathbb{R}$ is defined. Let Ω be an open subset of U such that*

(i) $0 \in \partial\Omega$,
(ii) $V(\boldsymbol{v}) = 0, \forall\, \boldsymbol{v} \in \partial\Omega \cap U$,
(iii) $V(\boldsymbol{v}) > 0$ *and* $\dot{V}(\boldsymbol{v}) > 0, \forall\, \boldsymbol{v} \in \Omega \cap U$.

Then the origin is an unstable equilibrium of (5.2).

The interpretation of this theorem is the following. An orbit $\boldsymbol{u}(t; \boldsymbol{u}_\circ)$ starting in $\Omega \cap U$, will never cross $\partial\Omega$ due to the properties (2) and (3) of the Chetaev function. From the second part of property (3) it now follows that $V(\boldsymbol{u}(t; \boldsymbol{u}_\circ))$ is increasing whenever $\boldsymbol{u}(t; \boldsymbol{u}_\circ)$ lies in $\Omega \cap U$. This orbit can not stay in $\partial\Omega \cap U$ due to the fact that U is open. Thus, $\boldsymbol{u}(t)$ moves away from the origin. Hence the origin is unstable.

6 Applications to the restricted problem

In this section we apply the theory of the previous sections to the relative equilibria of the restricted three-body problem.