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Contents

Preface

Corings and comodules are fundamental algebraic structures that can be thought of as both dualisations and generalisations of rings and modules. Corings were introduced by Sweedler in 1975 as a generalisation of coalgebras and as a means of presenting a semi-dual version of the Jacobson-Bourbaki Theorem, but their origin can be traced back to 1968 in the work of Jonah on cohomology of coalgebras in monoidal categories. In the late seventies they resurfaced under the name of bimodules over a category with a coalgebra structure, BOCSs for short, in the work of Rojter and Kleiner on algorithms for matrix problems. For a long time, essentially only two types of examples of corings truly generalising coalgebras were known – one associated to a ring extension, the other to a matrix problem. The latter example was also studied in the context of differential graded algebras and categories. This lack of examples hindered the full appreciation of the fundamental role of corings in algebra and obviously hampered their progress in general coring theory.

On the other hand, from the late seventies and throughout the eighties and nineties, various types of Hopf modules were studied. Initially these were typically modules and comodules of a common bialgebra or a Hopf algebra with some compatibility condition, but this evolved to modules of an algebra and comodules of a coalgebra with a compatibility condition controlled by a bialgebra. In fact, even the background bialgebra has been shown now to be redundant provided some relations between a coalgebra and an algebra are imposed in terms of an entwining. The progress and interest in such categories of modules were fuelled by the emergence of quantum groups and their application to physics, in particular gauge theory in terms of principal bundles and knot theory.

By the end of the last century, M. Takeuchi realised that the compatibility condition between an algebra and a coalgebra known as an entwining can be recast in terms of a coring. From this, it suddenly became apparent that various properties of Hopf modules, including entwined modules, can be understood and more neatly presented from the point of view of the associated coring. It also emerged, on the one hand, that coring theory is rich in many interesting examples and, on the other, that – based on the knowledge of Hopf-type modules – there is much more known about the general structure of corings than has been previously realised. It also turned out that corings might have a variety of unexpected and wide-ranging applications, to topics in noncommutative ring theory, category theory, Hopf algebras, differential graded algebras, and noncommutative geometry. In summary, corings appear to offer a new, exciting possibility for recasting known results in a unified general manner and for the development of ring and module theory from a completely different point of view.

As indicated above, corings can be viewed as generalisations of coalgebras, the latter an established and well-studied theory, in particular over fields. More precisely, a coalgebra over a commutative ring R can be defined as a coalgebra in the monoidal category of R-modules – a notion that is well known in general category theory. On the other hand, an A-coring is a coalgebra in the monoidal category of (A, A)-bimodules, where A is an arbitrary ring. With the emergence of quantum groups in the works of Drinfeld [110], Jimbo [135] and Woronowicz [214], new interest arose in the study of coalgebras, mainly those with additional structures such as bialgebras and Hopf algebras, because of their importance in various applications. In the majority of books on Hopf algebras and coalgebras, such as the now classic texts of Sweedler [45] and Abe [1] or in the more recent works of Montgomery [37] and Dăscălescu, Năstăsescu and Raianu [14] together with texts motivated by quantum group theory (e.g., Lusztig [30]; Majid [33, 34]; Chari and Pressley [11]; Shnider and Sternberg [43]; Kassel [25]; Klimyk and Schmüdgen [26]; Brown and Goodearl [7]), coalgebras are considered over fields. The vast variety of applications and new developments, in particular in ring and module theory, manifestly show that there is still a need for a better understanding of coalgebras over arbitrary commutative rings, as a preliminary step towards the theory of corings. Let us mention a few aspects of particular interest to the classical module and ring theory.

There are parts of module theory over algebras A that provide a perfect setting for the theory of comodules. Given any left A-module M, denote by $\sigma[M]$ the full subcategory of the category ${}_A\mathbf{M}$ of left A-modules that is subgenerated by M. This is the smallest Grothendieck subcategory of ${}_A\mathbf{M}$ containing M. Internal properties of $\sigma[M]$ strongly depend on the module properties of M, and there is a well-established theory that explores this relationship. Although, in contrast to ${}_A\mathbf{M}$, there need not be projectives in $\sigma[M]$, its Grothendieck property enables the use of techniques such as localisation and various homological methods in $\sigma[M]$. Consequently, one can gain a very good understanding of the *inner properties* of $\sigma[M]$. On the other hand, by definition, $\sigma[M]$ is closed under direct sums, submodules and factor modules in ${}_A\mathbf{M}$, and so it is a *hereditary pretorsion class* in ${}_A\mathbf{M}$. If $\sigma[M]$ is also closed under extensions in ${}_A\mathbf{M}$, it is a (*hereditary*) *torsion class*. Torsion theory then provides many characterisations of the *outer properties* of $\sigma[M]$, that is, the behaviour of $\sigma[M]$ as a subclass of ${}_A\mathbf{M}$.

Both the inner and outer properties of the categories of type $\sigma[M]$ are important in the study of coalgebras and comodules. If C is a coalgebra over a commutative ring R, then the dual C^* is an R-algebra and C is a left and right module over C^*. The link to the module theory mentioned above is provided by the basic observation that the category $\mathbf{M}^C$ of right C-comodules is subgenerated by C, and there is a faithful functor from $\mathbf{M}^C$

to the category ${}_{C^*}\mathbf{M}$ of left C^*-modules. Further properties of $\mathbf{M}^C$ and of this relationship depend on the properties of C as an R-module. First, if ${}_RC$ is (locally) projective, then $\mathbf{M}^C$ is the same as the category $\sigma[{}_{C^*}C]$, the full subcategory of ${}_{C^*}\mathbf{M}$ subgenerated by C. In this case, results for module categories of type $\sigma[M]$ can be transferred directly to comodules, with no new proofs required. This affords a deeper understanding of old results about comodules (for coalgebras over fields) and provides new proofs that readily apply to coalgebras over rings. For example, the inner properties of $\sigma[M]$ reveal the internal structure and decomposition properties of C, while the outer properties allow one to study when $\mathbf{M}^C$ is closed under extensions in ${}_{C^*}\mathbf{M}$. Second, if ${}_RC$ is flat, then, although $\mathbf{M}^C$ is no longer a full subcategory of ${}_{C^*}\mathbf{M}$, it is still an Abelian (Grothendieck) category, and the pattern of proofs for $\sigma[M]$ can often be followed literally to prove results for comodules. Third, in the absence of any condition on the R-module structure of ${}_RC$, the category $\mathbf{M}^C$ may lack kernels, and monomorphisms in $\mathbf{M}^C$ need not be injective maps. However, techniques from module theory may still be applied, albeit with more caution. For example, typical properties for Hopf algebras H do not need any restriction on ${}_RH$. Finally, it turns out that practically all of the traditional results, in the case of when R is a field, remain true over QF rings provided ${}_RC$ is flat.

Various properties of coalgebras over a commutative ring R extend to A-corings, provided one carefully addresses the noncommutativity of the base ring A in the latter case. For example, given an A-coring $\mathcal{C}$, one can consider three types of duals of $\mathcal{C}$, namely, the right A-module, the left A-module, or the (A, A)-bimodule dual. In each case one obtains a ring, but, since these rings are no longer isomorphic to each other, one has to carefully study relations between them. This then transfers to the study of categories of comodules of a coring and their relationship to various possible categories of modules over dual rings.

The aim of the present book is to give an introduction to the general theory of corings and to indicate their numerous applications. We would like to stress the role of corings as one of the most fundamental algebraic structures, which, in a certain sense, lie between module theory and category theory. From the former point of view, they give a unifying and general framework for rings and modules, while from the latter they are concrete realisations of adjoint pairs of functors or comonads. We start, however, with a description of the theory of coalgebras over commutative rings and their comodules. Therefore, the first part of the book should give the reader a feeling for the typical features of coalgebras over rings as opposed to fields, thus, in the first instance, filling a gap in the existing literature. It is also a preparation for the second part, which is intended to provide the reader with a reference to the wide tapestry of known results on the structure of corings, possible applications and developments. It is not our aim to give a complete picture of this rapidly developing theory. Instead we would like to indicate what is known and what can be done in this new, emerging field. Thus we provide an overview of known and, by now, standard results about corings scattered in the existing established literature. Furthermore, we outline various aspects of corings studied very recently by several authors in a number of published papers as well as preprints still awaiting formal publication. We believe, however, that a significant number of results included in this book are hereby published for the first time. It is our hope that the present book will become a reference and a starting point for further progress in this new exciting field. We also believe that the first part of the book, describing coalgebras over rings, may serve as a textbook for a graduate course on coalgebras and Hopf algebras for students who would like to specialise in algebra and ring theory.

A few words are in order to explain the structure of the book. The book is primarily intended for mathematicians working in ring and module theory and related subjects, such as Hopf algebras. We believe, however, that it will also be useful for (mathematically oriented) mathematical physicists, in particular those who work with quantum groups and noncommutative geometry. In the main text, we make passing references to how abstract constructions may be seen from their point of view. Moreover, the attention of noncommutative geometers should be drawn, in particular to the construction of connections in Section 29.

The book also assumes various levels of familiarity with coalgebras. The reader who is not familiar with coalgebras should start with Chapter 1. The reader who is familiar with coalgebras and Hopf algebras can proceed directly to Chapter 3 and return to preceding chapters when prompted. For the benefit of readers who are not very confident with the language of categories or with the structure of module categories, the main text is supplemented by an appendix in which we recall well-known facts about categories in general as well as module categories. This is done explicitly enough to provide a helpful guidance for the ideas employed in the main part of the text. Also included in the appendix are some new and less standard items that are used in the development of the theory of comodules in the main text.

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Notations

tw	the twist map $\mathsf{tw} : M \otimes_R N \to N \otimes_R M$, 40.1
$\operatorname{Ke} f$ $(\operatorname{Coke} f)$	the kernel (cokernel) of a linear map f
$\operatorname{Im} f$	the image of a map f
I, I_X	the identity morphism for an object X
A	algebra over a commutative ring R, 40.2
μ, μ_A	product of A as a map $A \otimes_R A \to A$, 40.2
ι, ι_A	the unit of A as a map $R \to A$, 40.2
$Z(A)$, $\operatorname{Jac}(A)$	the centre and the Jacobson radical of A
$\operatorname{Alg}_R(-,-)$	R-algebra maps
M_A $({}_AM)$	right (left) A-module M, 40.4
ϱ_M $({}_M\varrho)$	the A-action for a right (left) A-module M, 40.4
$\mathbf{M}_A$ $({}_A\mathbf{M})$	the category of right (left) A-modules, 40.4
$\operatorname{Hom}_A(-,-)$	homomorphisms of right A-modules, 40.4
${}_A\operatorname{Hom}(-,-)$	homomorphisms of left A-modules
${}_A\mathbf{M}_B$	the category of (A,B)-bimodules, 40.9
${}_A\operatorname{Hom}_B(-,-)$	homomorphisms of (A,B)-bimodules, 40.9
$\sigma[M]$	the full subcategory of $\mathbf{M}_A$ $({}_A\mathbf{M})$ of modules subgenerated by a module M, 41.1
C $(\mathcal{C})$	coalgebra over R (coring over A), 1.1, 17.1
Δ, Δ_C	the coproduct of C as map $C \to C \otimes_R C$, 1.1, 17.1
ε, ε_C	the counit of C as map $C \to R$, 1.1
$\underline{\Delta}$, $\underline{\Delta}_{\mathcal{C}}$	the coproduct of $\mathcal{C}$ as map $\mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}$, 17.1
$\underline{\varepsilon}$, $\underline{\varepsilon}_{\mathcal{C}}$	the counit of $\mathcal{C}$ as map $\mathcal{C} \to A$, 17.1
C^*	the dual (convolution) algebra of C, 1.3
$\mathcal{C}^*$, ${}^*\mathcal{C}$, ${}^*\mathcal{C}^*$	the right, left, and bi-dual algebras of $\mathcal{C}$, 17.8
ϱ^M $({}^M\varrho)$	the coaction of a right (left) comodule M, 3.1, 18.1
$\mathbf{M}^C$ $(\mathbf{M}^{\mathcal{C}})$	the category of right comodules over C $(\mathcal{C})$, 3.1, 18.1
$\operatorname{Hom}^C(-,-)$	the colinear maps of right C-comodules, 3.3
${}^{\mathcal{C}}\operatorname{Hom}(-,-)$	the colinear maps of left $\mathcal{C}$-comodules, 18.3
$\operatorname{End}^C(M)$	endomorphisms of a right C-comodule M, 3.12
${}^{\mathcal{C}}\operatorname{End}(M)$	endomorphisms of a left $\mathcal{C}$-comodule M, 18.12
${}^C\operatorname{Hom}^D(-,-)$	(C,D)-bicomodule maps, 11.1
$\operatorname{Rat}^{\mathcal{C}}(M)$	the rational comodule of a left ${}^*\mathcal{C}$-module M, 7.1, 20.1
${}^{\mathcal{C}}\mathbf{M}^{\mathcal{D}}$	the category of $(\mathcal{C},\mathcal{D})$-bicomodules, 22.1