Chapter 1

Coalgebras and comodules

Coalgebras and comodules are dualisations of algebras and modules. In this chapter we introduce the basic definitions and study several properties of these notions. The theory of coalgebras over fields and their comodules is well presented in various textbooks (e.g., Sweedler [45], Abe [1], Montgomery [37], Dăscălescu, Năstăsescu and Raianu [14]). Since the tensor product behaves differently over fields and rings, not all the results for coalgebras over fields can be extended to coalgebras over rings. Here we consider base rings from the very beginning, and part of our problems will be to find out which module properties of a coalgebra over a ring are necessary (and sufficient) to ensure the desired properties. In view of the main subject of this book, this chapter can be treated as a preliminary study towards corings. Also for this reason we almost solely concentrate on those properties of coalgebras and comodules that are important from the module theory point of view. The extra care paid to module properties of coalgebras will pay off in Chapter 3.

Throughout, R denotes a commutative and associative ring with a unit.

1 Coalgebras

Intuitively, a coalgebra over a ring can be understood as a dualisation of an algebra over a ring. Coalgebras by themselves are equally fundamental objects as are algebras. Although probably more difficult to understand at the beginning, they are often easier to handle than algebras. Readers with geometric intuition might like to think about algebras as functions on spaces and about coalgebras as objects that encode additional structure of such spaces (for example, group or monoid structure). The main aim of this section is to introduce and give examples of coalgebras and explain the (dual) relationship between algebras and coalgebras.

1.1. Coalgebras. An *R-coalgebra* is an R-module C with R-linear maps

$$\Delta : C \to C \otimes_R C \quad \text{and} \quad \varepsilon : C \to R,$$

called *(coassociative) coproduct* and *counit*, respectively, with the properties

$$(I_C \otimes \Delta) \circ \Delta = (\Delta \otimes I_C) \circ \Delta, \text{ and } (I_C \otimes \varepsilon) \circ \Delta = I_C = (\varepsilon \otimes I_C) \circ \Delta,$$

which can be expressed by commutativity of the diagrams

$$
\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes_R C \arrow[d, "I_C \otimes \Delta"] \\
C \otimes_R C \arrow[r, "\Delta \otimes I_C"] & C \otimes_R C \otimes_R C
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "I_C"] & C \otimes_R C \arrow[d, "\varepsilon \otimes I_C"] \\
C \otimes_R C \arrow[r, "I_C \otimes \varepsilon"'] & C\,.
\end{tikzcd}
$$

A coalgebra (C, Δ, ε) is said to be *cocommutative* if $\Delta = \mathsf{tw} \circ \Delta$, where

$$\mathsf{tw} : C \otimes_R C \to C \otimes_R C, \quad a \otimes b \mapsto b \otimes a,$$

is the twist map (cf. 40.1).

1.2. Sweedler's Σ-notation. For an elementwise description of the maps we use the Σ-*notation*, writing for $c \in C$

$$\Delta(c) = \sum_{i=1}^{k} c_i \otimes \tilde{c}_i = \sum c_{\underline{1}} \otimes c_{\underline{2}}.$$

The first version is more precise; the second version, introduced by Sweedler, turnes out to be very handy in explicit calculations. Notice that $c_{\underline{1}}$ and $c_{\underline{2}}$ do not represent single elements but families $c_1, \ldots, c_k$ and $\tilde{c}_1, \ldots, \tilde{c}_k$ of elements of C that are by no means uniquely determined. Properties of $c_{\underline{1}}$ can only be considered in context with $c_{\underline{2}}$. With this notation, the coassociativity of Δ is expressed by

$$\sum \Delta(c_{\underline{1}}) \otimes c_{\underline{2}} = \sum c_{\underline{1}\,\underline{1}} \otimes c_{\underline{1}\,\underline{2}} \otimes c_{\underline{2}} = \sum c_{\underline{1}} \otimes c_{\underline{2}\,\underline{1}} \otimes c_{\underline{2}\,\underline{2}} = \sum c_{\underline{1}} \otimes \Delta(c_{\underline{2}}),$$

and, hence, it is possible and convenient to shorten the notation by writing

$$
\begin{aligned}
(\Delta \otimes I_C)\Delta(c) = (I_C \otimes \Delta)\Delta(c) &= \sum c_{\underline{1}} \otimes c_{\underline{2}} \otimes c_{\underline{3}}, \\
(I_C \otimes I_C \otimes \Delta)(I_C \otimes \Delta)\Delta(c) &= \sum c_{\underline{1}} \otimes c_{\underline{2}} \otimes c_{\underline{3}} \otimes c_{\underline{4}},
\end{aligned}
$$

and so on. The conditions for the counit are described by

$$\sum \varepsilon(c_{\underline{1}})c_{\underline{2}} = c = \sum c_{\underline{1}}\varepsilon(c_{\underline{2}}).$$

Cocommutativity is equivalent to $\sum c_{\underline{1}} \otimes c_{\underline{2}} = \sum c_{\underline{2}} \otimes c_{\underline{1}}$.

R-coalgebras are closely related or dual to algebras. Indeed, the module of R-linear maps from a coalgebra C to any R-algebra is an R-algebra.

1.3. The algebra $\mathrm{Hom}_R(C, A)$. *For any R-linear map $\Delta : C \to C \otimes_R C$ and an R-algebra A, $\mathrm{Hom}_R(C, A)$ is an R-algebra by the* convolution product

$$f * g = \mu \circ (f \otimes g) \circ \Delta, \quad \textit{i.e.,} \quad f * g(c) = \sum f(c_{\underline{1}})g(c_{\underline{2}}),$$

for $f, g \in \mathrm{Hom}_R(C, A)$ and $c \in C$. Furthermore,

(1) *Δ is coassociative if and only if* $\mathrm{Hom}_R(C, A)$ *is an associative R-algebra, for any R-algebra A.*

(2) *C is cocommutative if and only if* $\mathrm{Hom}_R(C, A)$ *is a commutative R-algebra, for any commutative R-algebra A.*

(3) *C has a counit if and only if* $\mathrm{Hom}_R(C, A)$ *has a unit, for all R-algebras A with a unit.*

Proof. (1) Let $f, g, h \in \mathrm{Hom}_R(C, A)$ and consider the R-linear map

$$\tilde{\mu} : A \otimes_R A \otimes_R A \to A, \quad a_1 \otimes a_2 \otimes a_3 \mapsto a_1 a_2 a_3.$$

By definition, the products $(f * g) * h$ and $f * (g * h)$ in $\mathrm{Hom}_R(C, A)$ are the compositions of the maps

$$\begin{tikzcd}
 & C \otimes_R C \arrow[dr, "\Delta \otimes I_C"] & & & \\
C \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & C \otimes_R C \otimes_R C \arrow[r, "f \otimes g \otimes h"] & A \otimes_R A \otimes_R A \arrow[r, "\tilde{\mu}"] & A\,. \\
 & C \otimes_R C \arrow[ur, "I_C \otimes \Delta"'] & & &
\end{tikzcd}$$

It is obvious that coassociativity of Δ yields associativity of $\mathrm{Hom}_R(C, A)$.

To show the converse, we see from the above diagram that it suffices to prove that, (at least) for one associative algebra A and suitable $f, g, h \in \mathrm{Hom}_R(C, A)$, the composition $\tilde{\mu} \circ (f \otimes g \otimes h)$ is a monomorphism. So let $A = T(C)$, the tensor algebra of the R-module C (cf. 15.12), and $f = g = h$, the canonical mapping $C \to T(C)$. Then $\tilde{\mu} \circ (f \otimes g \otimes h)$ is just the embedding $C \otimes C \otimes C = T_3(C) \to T(C)$.

(2) If C is cocommutative and A is commutative,

$$f * g\,(c) = \sum f(c_{\underline{1}}) g(c_{\underline{2}}) = \sum g(c_{\underline{1}}) f(c_{\underline{2}}) = g * f\,(c),$$

so that $\mathrm{Hom}_R(C, A)$ is commutative. Conversely, assume that $\mathrm{Hom}_R(C, A)$ is commutative for any commutative A. Then

$$\mu \circ (f \otimes g)(\Delta(c)) = \mu \circ (f \otimes g)(\mathsf{tw} \circ \Delta(c)).$$

This implies $\Delta = \mathsf{tw} \circ \Delta$ provided we can find a commutative algebra A and $f, g \in \mathrm{Hom}_R(C, A)$ such that $\mu \circ (f \otimes g) : C \otimes_R C \to A$ is injective. For this take A to be the symmetric algebra $\mathcal{S}(C \oplus C)$ (see 15.13). For f and g we choose the mappings

$$C \to C \oplus C, \quad x \mapsto (x, 0), \qquad C \to C \oplus C, \quad x \mapsto (0, x),$$

composed with the canonical embedding $C \oplus C \to \mathcal{S}(C \oplus C)$.

With the canonical isomorphism $h : \mathcal{S}(C) \otimes \mathcal{S}(C) \to \mathcal{S}(C \oplus C)$ (see 15.13) and the embedding $\lambda : C \to \mathcal{S}(C)$, we form $h^{-1} \circ \mu \circ (f \otimes g) = \lambda \otimes \lambda$. Since $\lambda(C)$ is a direct summand of $\mathcal{S}(C)$, we obtain that $\lambda \otimes \lambda$ is injective and so $\mu \circ (f \otimes g)$ is injective.

(3) It is easy to check that the unit in $\mathrm{Hom}_R(C, A)$ is

$$C \xrightarrow{\varepsilon} R \xrightarrow{\iota} A, \quad c \mapsto \varepsilon(c)1_A.$$

For the converse, consider the R-module $A = R \oplus C$ and define a unital R-algebra

$$\mu : A \otimes_R A \to A, \quad (r, a) \otimes (s, b) \mapsto (rs, rb + as).$$

Suppose there is a unit element in $\mathrm{Hom}_R(C, A)$,

$$e : C \to A = R \oplus C, \quad c \mapsto (\varepsilon(c), \lambda(c)),$$

with R-linear maps $\varepsilon : C \to R$, $\lambda : C \to C$. Then, for $f : C \to A$, $c \mapsto (0, c)$, multiplication in $\mathrm{Hom}_R(C, A)$ yields

$$f * e : C \to A, \quad c \mapsto (0, (I_C \otimes \varepsilon) \circ \Delta(c)).$$

By assumption, $f = f * e$ and hence $I_C = (I_C \otimes \varepsilon) \circ \Delta$, one of the conditions for ε to be a counit. Similarly, the other condition is derived from $f = e * f$.

Clearly ε is the unit in $\mathrm{Hom}_R(C, R)$, showing the uniqueness of a counit for C. □

Note in particular that $C^* = \mathrm{Hom}_R(C, R)$ is an algebra with the convolution product known as the *dual* or *convolution algebra* of C.

Notation. From now on, C (usually) will denote a coassociative R-coalgebra (C, Δ, ε), and A will stand for an associative R-algebra with unit (A, μ, ι).

Many properties of coalgebras depend on properties of the base ring R. The base ring can be changed in the following way.

1.4. Scalar extension. *Let C be an R-coalgebra and S an associative commutative R-algebra with unit. Then $C \otimes_R S$ is an S-coalgebra with the coproduct*

$$\tilde{\Delta} : C \otimes_R S \xrightarrow{\Delta \otimes I_S} (C \otimes_R C) \otimes_R S \xrightarrow{\simeq} (C \otimes_R S) \otimes_S (C \otimes_R S)$$

and the counit $\varepsilon \otimes I_S : C \otimes_R S \to S$. If C is cocommutative, then $C \otimes_R S$ is cocommutative.

Proof. By definition, for any $c \otimes s \in C \otimes_R S$,

$$\tilde{\Delta}(c \otimes s) = \sum (c_{\underline{1}} \otimes 1_S) \otimes_S (c_{\underline{2}} \otimes s).$$

It is easily checked that $\tilde{\Delta}$ is coassociative. Moreover,

$$(\varepsilon \otimes I_S \otimes I_{C\otimes_R S}) \circ \tilde{\Delta}(c \otimes s) = \sum \varepsilon(c_{\underline{1}}) c_{\underline{2}} \otimes s = c \otimes s,$$

and similarly $(I_{C\otimes_R S} \otimes \varepsilon \otimes I_S) \circ \tilde{\Delta} = I_{C\otimes_R S}$ is shown.

Obviously cocommutativity of Δ implies cocommutativity of $\tilde{\Delta}$. □

To illustrate the notions introduced above we consider some examples.

1.5. R as a coalgebra. The ring R is (trivially) a coassociative, cocommutative coalgebra with the canonical isomorphism $R \to R \otimes_R R$ as coproduct and the identity map $R \to R$ as counit.

1.6. Free modules as coalgebras. Let F be a free R-module with basis $(f_\lambda)_\Lambda$, Λ any set. Then there is a unique R-linear map

$$\Delta : F \to F \otimes_R F, \quad f_\lambda \mapsto f_\lambda \otimes f_\lambda,$$

defining a coassociative and cocommutative coproduct on F. The counit is provided by the linear map $\varepsilon : F \to R$, $f_\lambda \longmapsto 1$.

1.7. Semigroup coalgebra. Let G be a semigroup. A coproduct and counit on the semigroup ring $R[G]$ can be defined by

$$\Delta_1 : R[G] \to R[G] \otimes_R R[G],\ g \mapsto g \otimes g, \quad \varepsilon_1 : R[G] \to R,\ g \mapsto 1.$$

If G has a unit e, then another possibility is

$$\Delta_2 : R[G] \to R[G] \otimes_R R[G], \quad g \mapsto \begin{cases} e \otimes e & \text{if } g = e, \\ g \otimes e + e \otimes g & \text{if } g \neq e. \end{cases}$$

$$\varepsilon_2 : R[G] \to R, \quad g \mapsto \begin{cases} 1 & \text{if } g = e, \\ 0 & \text{if } g \neq e. \end{cases}$$

Both Δ_1 and Δ_2 are coassociative and cocommutative.

1.8. Polynomial coalgebra. A coproduct and counit on the polynomial ring $R[X]$ can be defined as algebra homomorphisms by

$$\begin{array}{ll} \Delta_1 : R[X] \to R[X] \otimes_R R[X], & X^i \mapsto X^i \otimes X^i, \\ \varepsilon_1 : R[X] \to R, & X^i \mapsto 1, \quad i = 0, 1, 2, \ldots. \end{array}$$

or else by

$$\begin{array}{ll} \Delta_2 : R[X] \to R[X] \otimes_R R[X], & 1 \mapsto 1,\ X^i \mapsto (X \otimes 1 + 1 \otimes X)^i, \\ \varepsilon_2 : R[X] \to R, & 1 \mapsto 1,\ X^i \mapsto 0, \quad i = 1, 2, \ldots. \end{array}$$

Again, both Δ_1 and Δ_2 are coassociative and cocommutative.

1.9. Coalgebra of a projective module. Let P be a finitely generated projective R-module with dual basis $p_1, \ldots, p_n \in P$ and $\pi_1, \ldots, \pi_n \in P^*$. There is an isomorphism

$$P \otimes_R P^* \to \mathrm{End}_R(P), \quad p \otimes f \mapsto [a \mapsto f(a)p],$$

and on $P^* \otimes_R P$ the coproduct and counit are defined by

$$\Delta : P^* \otimes_R P \to (P^* \otimes_R P) \otimes_R (P^* \otimes_R P), \quad f \otimes p \mapsto \sum_i f \otimes p_i \otimes \pi_i \otimes p,$$

$$\varepsilon : P^* \otimes_R P \to R, \quad f \otimes p \mapsto f(p).$$

By properties of the dual basis,

$$(I_{P \otimes_R P^*} \otimes \varepsilon)\Delta(f \otimes p) = \sum_i f \otimes p_i \pi_i(p) = f \otimes p,$$

showing that ε is a counit, and coassociativity of Δ is proved by the equality

$$(I_{P \otimes_R P^*} \otimes \Delta)\Delta(f \otimes p) = \sum\nolimits_{i,j} f \otimes p_i \otimes \pi_i \otimes p_j \otimes \pi_j \otimes p = (\Delta \otimes I_{P \otimes_R P^*})\Delta(f \otimes p).$$

The dual algebra of $P^* \otimes_R P$ is (anti)isomorphic to $\mathrm{End}_R(P)$ by the bijective maps

$$(P^* \otimes_R P)^* = \mathrm{Hom}_R(P^* \otimes_R P, R) \simeq \mathrm{Hom}_R(P, P^{**}) \simeq \mathrm{End}_R(P),$$

which yield a ring isomorphism or anti-isomorphism, depending from which side the morphisms are acting.

For $P = R$ we obtain $R = R^*$, and $R^* \otimes_R R \simeq R$ is the trivial coalgebra. As a more interesting special case we may consider $P = R^n$. Then $P^* \otimes_R P$ can be identified with the matrix ring $M_n(R)$, and this leads to the

1.10. Matrix coalgebra. Let $\{e_{ij}\}_{1 \le i,j \le n}$ be the canonical R-basis for $M_n(R)$, and define the coproduct and counit

$$\Delta : M_n(R) \to M_n(R) \otimes_R M_n(R), \quad e_{ij} \mapsto \sum\nolimits_k e_{ik} \otimes e_{kj},$$

$$\varepsilon : M_n(R) \to R, \quad e_{ij} \mapsto \delta_{ij}\,.$$

The resulting coalgebra is called the *(n, n)-matrix coalgebra over R*, and we denote it by $M_n^c(R)$.

Notice that the matrix coalgebra may also be considered as a special case of a semigroup coalgebra in 1.7.

From a given coalgebra one can construct the

1.11. Opposite coalgebra. Let $\Delta : C \to C \otimes_R C$ define a coalgebra. Then

$$\Delta^{\mathsf{tw}} : C \xrightarrow{\Delta} C \otimes_R C \xrightarrow{\mathsf{tw}} C \otimes_R C, \quad c \mapsto \sum c_{\underline{2}} \otimes c_{\underline{1}},$$

where tw is the twist map, defines a new coalgebra structure on C known as the *opposite* coalgebra with the same counit. The opposite coalgebra is denoted by C^{cop}. Note that a coalgebra C is cocommutative if and only if C coincides with its opposite coalgebra (i.e., $\Delta = \Delta^{\mathsf{tw}}$).

1.12. Duals of algebras. Let (A, μ, ι) be an R-algebra and assume ${}_RA$ to be finitely generated and projective. Then there is an isomorphism

$$A^* \otimes_R A^* \to (A \otimes_R A)^*, \quad f \otimes g \mapsto [a \otimes b \mapsto f(a)g(b)],$$

and the functor $\mathrm{Hom}_R(-, R) = (-)^*$ yields a coproduct

$$\mu^* : A^* \to (A \otimes_R A)^* \simeq A^* \otimes_R A^*$$

and a counit (as the dual of the unit of A)

$$\varepsilon := \iota^* : A^* \to R, \quad f \mapsto f(1_A).$$

This makes A^* an R-coalgebra that is cocommutative provided μ is commutative. If ${}_RA$ is not finitely generated and projective, the above construction does not work. However, under certain conditions the *finite dual* of A has a coalgebra structure (see 5.7).

Further examples of coalgebras are the tensor algebra 15.12, the symmetric algebra 15.13, and the exterior algebra 15.14 of any R-module, and the enveloping algebra of any Lie algebra.

1.13. Exercises

Let $M_n^c(R)$ be a matrix coalgebra with basis $\{e_{ij}\}_{1\leq i,j\leq n}$ (see 1.10). Prove that the dual algebra $M_n^c(R)^*$ is an (n, n)-matrix algebra.

(Hint: Consider the basis of M^* dual to $\{e_{ij}\}_{1\leq i,j\leq n}$.)

References. Abuhlail, Gómez-Torrecillas and Wisbauer [50]; Bourbaki [5]; Sweedler [45]; Wisbauer [210].

2 Coalgebra morphisms

To discuss coalgebras formally, one would like to understand not only isolated coalgebras, but also coalgebras in relation to other coalgebras. In a word, one would like to view coalgebras as objects in a category.[1] For this one needs the notion of a *coalgebra morphism*. Such a morphism can be defined as an R-linear map between coalgebras that respects the coalgebra structures (coproducts and counits). The idea behind this definition is of course borrowed from the idea of an algebra morphism as a map respecting the algebra structures. Once such morphisms are introduced, relationships between coalgebras can be studied. In particular, we can introduce the notions of a *subcoalgebra* and a *quotient coalgebra*. These are the topics of the present section.

2.1. Coalgebra morphisms. Given R-coalgebras C and C', an R-linear map $f : C \to C'$ is said to be a *coalgebra morphism* provided the diagrams

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta"'] & C' \arrow[d, "\Delta'"] \\ C \otimes_R C \arrow[r, "f\otimes f"] & C' \otimes_R C' , \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "f"] \arrow[dr, "\varepsilon"'] & C' \arrow[d, "\varepsilon'"] \\ & R \end{tikzcd}$$

are commutative. Explicitly, this means that

$$\Delta' \circ f = (f \otimes f) \circ \Delta, \quad \text{and} \quad \varepsilon' \circ f = \varepsilon,$$

that is, for all $c \in C$,

$$\sum f(c_{\underline{1}}) \otimes f(c_{\underline{2}}) = \sum f(c)_{\underline{1}} \otimes f(c)_{\underline{2}}, \quad \text{and} \quad \varepsilon'(f(c)) = \varepsilon(c).$$

Given an R-coalgebra C and an S-coalgebra D, where S is a commutative ring, a *coalgebra morphism* between C and D is defined as a pair (α, γ) consisting of a ring morphism $\alpha : R \to S$ and an R-linear map $\gamma : C \to D$ such that

$$\gamma' : C \otimes_R S \to D, \quad c \otimes s \mapsto \gamma(c)s,$$

is an S-coalgebra morphism. Here we consider D as an R-module (induced by α) and $C \otimes_R S$ is the scalar extension of C (see 1.4).

As shown in 1.3, for an R-algebra A, the contravariant functor $\mathrm{Hom}_R(-, A)$ turns coalgebras to algebras. It also turns coalgebra morphisms into algebra morphisms.

[1]The reader not familiar with category theory is referred to the Appendix, §38.

2.2. Duals of coalgebra morphisms. *For R-coalgebras C and C', an R-linear map $f : C \to C'$ is a coalgebra morphism if and only if*

$$\mathrm{Hom}(f, A) : \mathrm{Hom}_R(C', A) \to \mathrm{Hom}_R(C, A)$$

is an algebra morphism, for any R-algebra A.

Proof. Let f be a coalgebra morphism. Putting $f^* = \mathrm{Hom}_R(f, A)$, we compute for $g, h \in \mathrm{Hom}_R(C', A)$

$$\begin{aligned} f^*(g * h) &= \mu \circ (g \otimes h) \circ \Delta' \circ f = \mu \circ (g \otimes h) \circ (f \otimes f) \circ \Delta \\ &= (g \circ f) * (h \circ f) \ = \ f^*(g) * f^*(h). \end{aligned}$$

To show the converse, assume that f^* is an algebra morphism, that is,

$$\mu \circ (g \otimes h) \circ \Delta' \circ f = \mu \circ (g \otimes h) \circ (f \otimes f) \circ \Delta,$$

for any R-algebra A and $g, h \in \mathrm{Hom}_R(C', A)$. Choose A to be the tensor algebra $T(C)$ of the R-module C and choose g, h to be the canonical embedding $C \to T(C)$ (see 15.12). Then $\mu \circ (g \otimes h)$ is just the embedding $C \otimes_R C \to T_2(C) \to T(C)$, and the above equality implies

$$\Delta' \circ f = (f \otimes f) \circ \Delta,$$

showing that f is a coalgebra morphism. □

2.3. Coideals. The problem of determining which R-submodules of C are kernels of a coalgebra map $f : C \to C'$ is related to the problem of describing the kernel of $f \otimes f$ (in the category of R-modules $\mathbf{M}_R$). If f is surjective, we know that $\mathrm{Ke}\,(f \otimes f)$ is the sum of the canonical images of $\mathrm{Ke}\, f \otimes_R C$ and $C \otimes_R \mathrm{Ke}\, f$ in $C \otimes_R C$ (see 40.15). This suggests the following definition.

The kernel of a surjective coalgebra morphism $f : C \to C'$ is called a *coideal* of C.

2.4. Properties of coideals. *For an R-submodule $K \subset C$ and the canonical projection $p : C \to C/K$, the following are equivalent:*

(a) K is a coideal;

(b) C/K is a coalgebra and p is a coalgebra morphism;

(c) $\Delta(K) \subset \mathrm{Ke}\,(p \otimes p)$ and $\varepsilon(K) = 0$.

If $K \subset C$ is C-pure, then (c) is equivalent to:

(d) $\Delta(K) \subset C \otimes_R K + K \otimes_R C$ and $\varepsilon(K) = 0$.

If (a) holds, then C/K is cocommutative provided C is also.

Proof. (a) ⇔ (b) is obvious.

(b) ⇒ (c) There is a commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & C & \xrightarrow{p} & C/K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \bar{\Delta}} & & \\ 0 & \longrightarrow & \mathrm{Ke}\,(p\otimes p) & \longrightarrow & C\otimes_R C & \xrightarrow{p\otimes p} & C/K\otimes_R C/K & \longrightarrow & 0, \end{array}$$

where commutativity of the right square implies the existence of a morphism $K \to \mathrm{Ke}\,(p\otimes p)$, thus showing $\Delta(K) \subset \mathrm{Ke}\,(p\otimes p)$. For the counit $\bar{\varepsilon} : C/K \to R$ of C/K, $\bar{\varepsilon}\circ p = \varepsilon$ and hence $\varepsilon(K) = 0$

(c) ⇒ (b) Under the given conditions, the left-hand square in the above diagram is commutative and the cokernel property of p implies the existence of $\bar{\Delta}$. This makes C/K a coalgebra with the properties required.

(c) ⇔ (d) If $K \subset C$ is C-pure, $\mathrm{Ke}\,(p\otimes p) = C\otimes_R K + K\otimes_R C$. □

2.5. Factorisation theorem. *Let $f : C \to C'$ be a morphism of R-coalgebras. If $K \subset C$ is a coideal and $K \subset \mathrm{Ke}\, f$, then there is a commutative diagram of coalgebra morphisms*

$$\begin{array}{ccc} C & \xrightarrow{p} & C/K \\ & \searrow^{f} & \downarrow{\scriptstyle \bar{f}} \\ & & C' . \end{array}$$

Proof. Denote by $\bar{f} : C/K \to C'$ the R-module factorisation of $f : C \to C'$. It is easy to show that the diagram

$$\begin{array}{ccc} C/K & \xrightarrow{\bar{f}} & C' \\ {\scriptstyle \bar{\Delta}}\downarrow & & \downarrow{\scriptstyle \Delta'} \\ C/K\otimes_R C/K & \xrightarrow{\bar{f}\otimes\bar{f}} & C'\otimes_R C' \end{array}$$

is commutative. This means that $\bar{f}$ is a coalgebra morphism. □

2.6. The counit as a coalgebra morphism. *View R as a trivial R-coalgebra as in 1.5. Then, for any R-coalgebra C,*

(1) ε is a coalgebra morphism;

(2) if ε is surjective, then $\mathrm{Ke}\,\varepsilon$ is a coideal.

Proof. (1) Consider the diagram

$$\begin{array}{ccc} C & \xrightarrow{\varepsilon} & R \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \simeq} \\ C\otimes_R C & \xrightarrow{\varepsilon\otimes\varepsilon} & R\otimes_R R \end{array} \qquad \begin{array}{ccc} c & \longmapsto & \varepsilon(c) \\ \downarrow & & \downarrow \\ \sum c_{\underline{1}}\otimes c_{\underline{2}} & \longmapsto & \sum \varepsilon(c_{\underline{1}})\otimes\varepsilon(c_{\underline{2}}) . \end{array}$$