Introduction

A history of the D(2)-problem

The problem with which this book is concerned arose from the attempt, during the 1960s, to classify compact manifolds by means of 'surgery' [7], [48], [71], [73]. Developing further the techniques of Thom [64], Wallace [75], [76], [77], Milnor [43], and Smale [52], a movement led notably by W. Browder, S. P. Novikov, and C. T. C. Wall made a systematic effort to understand compact manifolds in terms of homotopy theory which, by that time, was already a mature subject, with its own highly developed literature and was considered, in practice, at least under the simplifying restriction of simple connectivity, to be effectively computable [8].

Wall's particular contribution to manifold theory was to consider surgery problems in which the fundamental group is non-trivial. Perhaps one should point out that by allowing *all* finitely presented fundamental groups, one automatically turns a computable theory into a noncomputable one [6], [47]. However, even if we restrict our attention to fundamental groups which are familiar, the extent to which the resulting theory is computable is problematic. It really depends upon what is meant by 'familiar', and how well one understands the group under consideration. When describing groups by means of generators and relations, there are easily stated questions which one can ask of very familiar and otherwise tractable finite groups which at present seem completely beyond our ability to answer.

In connection with this general attack, Wall wrote two papers which merit special attention. The first of these, 'Poincaré Complexes I' [72], gives general homotopical conditions which must be satisfied by any space before it can be transformed, by surgery, into a manifold.

The second, 'Finiteness conditions for CW complexes' [68] (and despite the earlier publication date, it does seem to come later in historical development), straddles the boundary between surgery and a more general attempt to describe all homotopy theory in terms of pure algebra, namely 'algebraic' or 'combinatorial' homotopy theory [80]. Wall's aim in this paper is to formulate general conditions which guarantee that a given space will be homotopically equivalent to one with certain properties. In particular, he asks what conditions it is necessary to impose before a space can be homotopy equivalent to one of dimension $\leq n$.

The obvious first condition that one looks for is that homology groups should vanish in dimensions $> n$. This is clearly a necessary condition. However, homology alone is a notoriously bad indicator of dimension as the following 'Moore space' example shows.

Let m be a positive integer; the Moore space $M(m, n)$ is formed from the n-sphere by attaching an $(n+1)$-cell by an attaching map $S^n \to S^n$ of degree m. Then $M(m, n)$ has dimension $n+1$. However, computing integral homology gives

$$H_k(M(m,n);\mathbf{Z}) = \begin{cases} \mathbf{Z/mZ} & k = n \\ 0 & n < k \end{cases}$$

which falsely indicates the dimension as dim $= n$. In fact, the accurate indicator of dimension is integral cohomology, and in this case we get

$$H^k(M(m,n);\mathbf{Z}) = \begin{cases} \mathbf{Z/mZ} & k = n+1 \\ 0 & n+1 < k \end{cases}$$

giving the correct answer $\dim(M(m,n)) = n+1$.

If $\tilde{X}$ denotes the universal covering of X, the assumption that $H_k(\tilde{X};\mathbf{Z}) = 0$ for all $n < k$ is enough to guarantee that X is equivalent to a space of dimension $\leq n+1$, but not necessarily of dimension $\leq n$. Therefore, we may pose the problem in the following form:

D(n)-problem: Let X be a complex of geometrical dimension $n+1$. What further conditions are necessary and sufficient for X to be homotopy equivalent to a complex of dimension n?

In the simply connected case, Milnor (unpublished) had previously shown that the necessary condition $H^{n+1}(X;\mathbf{Z}) = 0$, abstracted from the Moore space example, was already sufficient. In the non-simply connected case, clearly, one still requires $H_{n+1}(\tilde{X};\mathbf{Z}) = 0$. That being assumed, Wall showed that in dimensions $n \geq 3$, the additional condition, both necessary and sufficient, is the obvious generalization from the simply connected case, namely that $H^{n+1}(X;\mathcal{B}) = 0$ should hold for all coefficient bundles $\mathcal{B}$.

Wall also gave a 'formal' solution to the D(1)-problem, at the cost of using nonabelian sheaves $\mathcal{B}$. To an extent this was unsatisfactory, and the one-dimensional case was not cleared up completely until the Stallings–Swan proof that groups of cohomological dimension one are free [54], [61]. This left only the two-dimensional case, which we state in the following form:

D(2)-problem: Let X be a finite connected cell complex of geometrical dimension 3, and suppose that

$$H_3(\tilde{X}; \mathbf{Z}) = H^3(X; \mathcal{B}) = 0$$

for all coefficient systems $\mathcal{B}$ on X. Is it true that X is homotopy equivalent to a finite complex of dimension 2?

We shall say that a 3-complex X is *cohomologically two-dimensional* when these two conditions are satisfied. We note, and shall do so again in detail at the appropriate point, that for finite fundamental groups G, the condition $H_3(\tilde{X}; \mathbf{Z}) = 0$ is redundant, since it is implied, using the Eckmann–Shapiro Lemma, by the condition $H^3(X; \mathbf{F}) = 0$ where $\mathbf{F}$ is the standard coefficient bundle on X with fibre $\mathbf{Z}[G]$.

We have chosen to ask the question with the restriction that X be a finite complex. One can, of course, relax this condition; one can also ask a similar question phrased in terms of collapses and expansions [70]. Neither, however, will be pursued here.

The first thing to observe is that the D(2)-problem is parametrized by the fundamental group. Each finitely presented group G has its own D(2)-problem; we say that the D(2)-property *holds for* G when the above question is answered in the affirmative, and likewise *fails for* G when there is a finite 3-complex X_G with $\pi_1(X_G) \cong G$ which answers the above question in the negative.

The D(2)-problem arises in a completely natural way once one attempts, as Wall did in [72], to find a normal form for Poincaré complexes. To see how, consider a smooth closed connected n-manifold M^n, and let M_0 be the bounded manifold obtained by removing an open disc; by Morse Theory, it is easy to see that M_0 contracts on to a subcomplex of dimension $\leq n-1$; in particular, M admits a cellular decomposition with a single-top dimensional cell. In [72] Wall attempted to mimic this construction in the context of Poincaré complexes. He showed that, as a consequence of Poincaré Duality, a finite Poincaré complex M of dimension $n + 1 \geq 4$ has, up to homotopy, a representation in the form

$$(*) \qquad M = X \cup_\alpha e^{n+1}$$

where X is finite complex satisfying $H_{n+1}(\tilde{X}; \mathbf{Z}) = H^{n+1}(X; \mathcal{B}) = 0$ for all coefficient systems $\mathcal{B}$. Wall showed, in [68], that the case $n \geq 3$ sucessfully mimics the situation for manifolds in that X is equivalent to an n-complex. When $n = 1$, it follows from the Stallings–Swan Theorem [54], [61] that X is homotopy equivalent to a one-dimensional complex. It is only in the case $n = 2$, corresponding to a Poincaré 3-complex, that we still do not know the general answer.

The two-dimensional realization problem

In the world of low-dimensional topology, there is another, older, problem which can be posed independently. If K is a finite 2-complex with $\pi_1(K) = G$ one obtains an exact sequence of $\mathbf{Z}[G]$-modules of the form

$$0 \to \pi_2(K) \to C_2(K) \stackrel{\partial_2}{\to} C_1(K) \stackrel{\partial_1}{\to} C_0(K) \stackrel{\epsilon}{\to} \mathbf{Z} \to 0$$

where $\pi_2(K)$ is the second homotopy group of K, and $C_n(K) = H_n(\tilde{K}^{(n)}, \tilde{K}^{(n-1)})$ is the group of cellular n-chains in the universal cover of K. Since each $C_n(K)$ is a free module over $\mathbf{Z}[G]$, this suggests that we take, as algebraic models for geometric 2-complexes, arbitrary exact sequences of the form

$$0 \to J \to F_2 \stackrel{\partial_2}{\to} F_1 \stackrel{\partial_1}{\to} F_0 \stackrel{\epsilon}{\to} \mathbf{Z} \to 0$$

where F_i is a finitely generated free (or, more generally, stably free) module over $\mathbf{Z}[G]$. Such objects are called algebraic 2-complexes over G, and form a category denoted by $\mathbf{Alg}_G$.

In fact, the correspondence $K \mapsto C_*(K)$ gives a faithful representation of the two-dimensional geometric homotopy relation in the algebraic homotopy category determined by $\mathbf{Alg}_G$. That is, when K, L are finite geometrical 2-complexes with $\pi_1(K) = \pi_1(L) = G$, then $K \simeq L \iff C_*(K) \simeq C_*(L)$. This has been known since the time of Whitehead [80], and perhaps even from the time of Tietze [66]. Nevertheless, it is still difficult to find explicitly in the literature in this form, and we prove it directly in Chapter 9. There is now an obvious question.

Realization Problem: Let G be a finitely presented group. Is every algebraic 2-complex

$$\left(0 \to J \to F_2 \stackrel{\partial_2}{\to} F_1 \stackrel{\partial_1}{\to} F_0 \stackrel{\epsilon}{\to} \mathbf{Z} \to 0\right) \in \mathbf{Alg}_G$$

geometrically realizable; that is, homotopy equivalent in the algebraic sense, to a complex of the form $C_*(K)$ where K is a finite 2-complex?

Statement of results

Firstly we show (see [28]) that for *finite fundamental groups* G, the D(2)-problem is entirely equivalent to the Realization Problem; that is:

Theorem I: When G is a finite group, the D(2)-property holds for G if and only if each algebraic 2-complex over G is geometrically realizable.

Theorem I will also be referred to as the Realization Theorem. The proof given in the text uses techniques which are specific to finite groups and does not generalize immediately to infinite fundamental groups. In particular, we make frequent use of the fact that, over a finite group, projective modules are injective relative to the class of $\mathbf{Z}[G]$-lattices, a statement which is known to be false for even the most elementary of all infinite groups, namely the infinite cyclic group. This is not to say that the Realization Theorem, as stated, does not hold more widely. In Appendix B, we give a proof which is valid for all finitely presented groups which satisfy an additional homological finiteness condition, the so-called FL(3) condition.

Having reduced the D(2)-problem to the Realization Problem, to make progress we must now pursue the problem of realizing homotopy types of algebraic 2-complexes by geometric 2-complexes. Here we are helped by two specific technical advances which, considered together, render our task easier, at least for homotopy types over a finite fundamental group.

The first is Yoneda's Theory of module extensions [34], [82]. This was in essence known to Whitehead, as can be seen from [35]. It is rather the modern version of Yoneda Theory, expressed in terms of stable modules and derived categories, implicit in the original, but incompletely realized, that the author has found so useful. Since the systematic use of stable modules is such an obvious feature of the exposition, some words of explanation are perhaps in order.

For a module M over a ring Λ, the *stable module* $[M]$ is the equivalence class of M under the equivalence relation generated by the stabilization operation $M \mapsto M \oplus \Lambda$. We shall also need to consider a more general stability, here called *hyper-stability*,* namely $M \mapsto M \oplus P$, where P is an arbitrary finitely generated projective module.

For much of the time we work, not in the category of modules over $\mathbf{Z}[G]$, but rather in the 'derived module category'. This is the quotient category obtained by equating 'projective = 0'. The objects in this category can be equated with hyper-stable classes of modules.

To make a comparison with a simpler case, Carlson's book [11] considers modular representation theory systematically from this point of view, and his

* Mac Lane calls this notion 'projective equivalence' ([34], p. 101, Exercise 2).

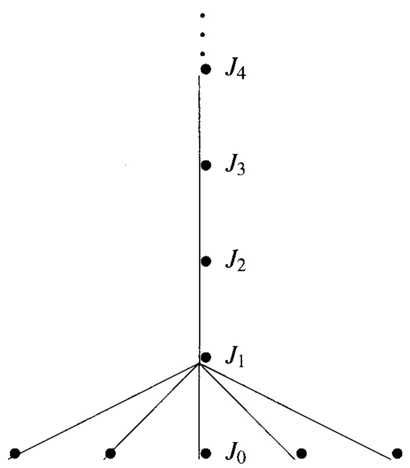
elegant account was extremely useful in the initial formulation of ideas. There all projectives are free, and objects in the derived category are indistinguishable from stable modules.

The derived category gives an *objective* form to the original Eilenberg–MacLane conception of homological algebra. Whereas they worked with 'derived functors', we work with 'derived objects'; cohomology as the derived functor of Hom is obtained simply by applying Hom to the appropriate derived objects.

Systematic use of derived objects confers some specific advantages. In the context of the cohomology of finite groups, it is 'well known and obvious' when pointed out, but has emerged from the collective subconscious only comparatively recently, [22], that cohomology functors are both representable and co-representable, in the technical sense of Yoneda's Lemma. Perhaps, given the minute analysis to which two generations have subjected the foundations of the subject, this is still less obvious than it should be. This 'geometrization' of cohomology allows a significant degree of control over the D(2)-problem, and a subsidiary aim of this book, carried out in Chapter 4, is to give an account of relative homological algebra from this point of view.

The second advance, Swan–Jacobinski cancellation theory, deals with the extent to which one can reverse the stabilization operation $M \mapsto M \oplus \mathbf{Z}[G]$. It enables us to assemble the set of all possible homotopy groups of two-dimensional complexes with given fundamental group into a tree, $\Omega_3(\mathbf{Z})$, of the following sort

$\Omega_3(\mathbf{Z})$



Here the vertices are modules, and one draws a line (upwards) joining two modules so $J_m \to J_m \oplus \mathbf{Z}[G] = J_{m+1}$. This idea of tree representation goes back to Dyer and Sieradski [16]. Homotopy types with π_2 at the bottom level are called *minimal*. We say that G has the *realization property* when all algebraic 2-complexes are geometrically realizable. Our second result (again see [28]) says that it is enough to realize homotopy types at the minimal level:

Theorem II: The finite group G has the realization property if and only if all minimal algebraic 2-complexes are realizable.

This is a general condition on homotopy types. Our third result gives a criterion for realization in terms of the second homotopy group rather than a complete homotopy type. For any such module J, there is an homomorphism of groups

$$\nu^J : \mathrm{Aut}_{\mathbf{Z}[G]}(J) \to \mathrm{Aut}_{\mathcal{D}\mathrm{er}}(J)$$

from the automorphism group in the module category to the automorphism group in the derived category. Moreover, $\mathrm{Im}(\nu^J)$ is contained in a certain subgroup of $\mathrm{Aut}_{\mathcal{D}\mathrm{er}}(J)$ namely the kernel of the Swan map

$$S : \mathrm{Aut}_{\mathcal{D}\mathrm{er}}(J) \to \widetilde{K}_0(\mathbf{Z}[G])$$

where $\widetilde{K}_0(\mathbf{Z}[G])$ is the projective class group. J is said to be *full* when $\mathrm{Im}(\nu^J) = \mathrm{Ker}(S)$. The module J is said to be *realizable* whenever J occurs in the form $J = \pi_2(K)$ for some finite 2-complex K with $\pi_1(K) = G$. Then we also have:

Theorem III: If each minimal module $J \in \Omega_3(\mathbf{Z})$ is both realizable and full, then G has the realization property.

Describing minimal homotopy types

For a given finite group G, obtaining an affirmative answer to the D(2)-problem involves two quite distinct steps. The first consists in giving a precise description of minimal two-dimensional algebraic homotopy types. Although possibly very difficult to implement in practice, in any particular case this step is amenable to procedures which are, in theory at least, effective. The second, for which no general procedure can be expected and which is still unsolved even for some very familiar and quite small groups, consists in determining enough minimal presentations of the group G under discussion to realize the minimal chain homotopy types.

In respect of the classification of the homotopy types of algebraic 2-complexes, the groups we find it easiest to deal with are the the finite groups of cohomological period 4. Chapter 7 is devoted to a brief exposition of what is known about them. As we point out in Chapter 11, groups of period 4 arise naturally in any discussion of Poincaré 3-complexes. For these groups, the general programme becomes much simpler. In particular, the homotopy types of minimal algebraic 2-complexes can be parametrized by more familiar objects. In Chapter 9, we prove the following classification result:

Theorem IV: Let G be a finite group of free period 4. Then there is a 1–1 correspondence

$$\widehat{\mathbf{Alg}}_G \longleftrightarrow SF(\mathbf{Z}[G])$$

where $\widehat{\mathbf{Alg}}_G$ is the set of two-dimensional algebraic homotopy types over G and $SF(\mathbf{Z}[G])$ is the set of isomorphism classes of stably free modules over $\mathbf{Z}[G]$.

Despite the intractability of the general problem of minimal group presentations mentioned above, for some groups enough is known to allow a complete solution of the D(2)-problem.

At present, Theorem IV gives the most hopeful candidate for a counterexample to the D(2)-property, for when $\mathbf{Z}[G]$ admits non-trivial stably free modules there are 'exotic' minimal 2-complexes which are not, as yet, known to be geometrically realizable. As the author has shown in [29], this occurs in the case of quaternion groups of high enough order. In fact, it follows from Swan's calculations [63] that the smallest example of this type is $Q(24)$, the quaternion group of order 24.

We point out that a special case of this classification, for the subclass of finite groups which are fundamental groups of closed 3-manifolds and which also possess the cancellation property for free modules, has also been obtained by Beyl, Latiolais and Waller [5] using a rather more geometric approach. In this case, there are no exotic minimal 2-complexes.

We note that the classification for some dihedral groups, where again there are no surprises, can be also be done directly by writing down explicit homotopy equivalences [27]. By contrast with [5], a theorem of Milnor [42] shows that dihedral groups are *not* fundamental groups of closed 3-manifolds.

It is known that if

$$\mathcal{G} = \langle X_1, \ldots, X_g; \quad W_1, \ldots, W_r \rangle$$

is a presentation of a finite group G then $g \leq r$. In the case where $g = r$, the presentation is said to be *balanced*. In terms of the D(2)-problems, finite groups which possess balanced presentations are 'smallest possible'. In connection with the problem of finding Poincaré 3-complexes of standard form, that is, having a single 3-cell, we obtain:

Theorem V: Let G be a finite group; then G has a standard Poincaré 3-form if and only if there is a finite presentation $\mathcal{G}$ of G for which $\pi_2(\mathcal{G}) \cong \mathbf{I}^*(G)$, the dual module of the augmentation ideal. Moreover, G then necessarily has free period 4, and the presentation $\mathcal{G}$ is automatically balanced.

For certain finite groups of period 4, the connection between the standard form problem and the D(2)-problem is one of equivalence.

Theorem VI: Let G be a finite group which admits a free resolution of period 4; if G has the free cancellation property then

$$G \text{ satisfies the D(2)-property} \iff G \text{ admits a balanced presentation.}$$

Earlier work on the algebraic classification problem

The first significant classification result of the type considered here is that of W. H. Cockroft and R. G. Swan [13], which classifies algebraic 2-complexes over a finite cyclic group. Taken in conjunction with Theorem I this is enough to answer the D(2)-question for finite cyclic groups in the affirmative. A general attack upon this question for finite fundamental groups was undertaken in a series of papers in the 1970s by M. N. Dyer and A. J. Sieradski. Although it does relate directly to our main concerns, one should also mention the contributions of Metzler on 2-complexes with finite abelian fundamental group [39].

However, although again we make very little direct appeal to it, without any doubt the next advance of real significance in the theory of two-dimensional homotopy was made by W. Browning in the late 1970s and early 1980s. The circumstances of Browning's career challenge the self-congratulatory assumptions on which the modern world is apt to reproach the past. In fact, Browning did not publish his results, and his work, available only in the form of his thesis [8], and some ETH pre-prints, languished in semi-obscurity for a number of years, before being accorded some of the recognition which it deserves. Happily, there is now a growing literature dealing with various aspects and generalizations of Browning's work; see for example, [20], [21], [33].

The principal result of Browning's thesis is his Stability Theorem which is a generalization from modules to chain complexes (essentially it is a complete re-proof) of the Swan–Jacobinski Theorem. Our realization results manage to avoid the technicalities of Browning's approach, though we shall review it briefly at the appropriate point in Chapter 9.

Browning's later work requires a restriction (the Eichler condition, see Chapter 3) which prevents it being applicable, except in trivial cases, to finite groups of period 4. Theorems III and IV by-pass the non-cancellation phenomena which arise when the Eichler condition fails, and are, in a sense, complementary to Browning's approach.

About this book

This book is essentially in three parts, with two appendices. In Chapters 1–3 we summarize those aspects of module theory and group representation theory that we shall need. Here the one really substantial piece of mathematics which we use systematically without any indication of proof is the Swan–Jacobinski Cancellation Theorem [25], [62]; for a proof we refer the reader to the definitive account of Curtis and Reiner [14] (vol. 2, Section 51).

Chapters 4–7 concentrate on group cohomology and module extension theory. Chapter 4 gives a systematic treatment of Yoneda Theory for our purposes; Chapters 5 and 6 specialize the general treatment to modules over group rings; Chapter 6 is particularly important since it contains the detailed classification theory by 'k-invariants' that we shall use systematically. Chapter 7 is devoted to the structure and classification of groups of periodic cohomology, which form the body of examples investigated later.

In Chapters 8–11 we consider algebraic and two-dimensional geometric homotopy theory in relation to the Realization and D(2)-problems.

Finally, in two appendices we consider briefly how the some of the arguments generalize to infinite fundamental groups. In Appendix A we show that the D(2)-property holds for finitely generated (non-abelian) free groups. In Appendix B we show that the Realization Theorem holds for finitely presented groups of type FL(3).

This work had its origin in a sequence of computations on module extensions over finite groups, with the general intention of investigating the structure theory of Poincaré 3-complexes. Begun in the Autumn of 1996, they were undertaken at first somewhat in the spirit of a diversion, for the sake of seeing what came out. However unsystematic, they nevertheless led, in short order to a perspective on the D(2)-problem which encouraged real hope of progress. Proofs of Theorems I and II followed shortly, and were announced at the British Topology Conference at Oxford in April 1997.

A number of cases of the D(2)-problem were solved on an *ad hoc* case-by-case basis in the spring and early summer of 1997. A more systematic attack required additional insight. Some of this was immediately forthcoming. At the Oxford meeting, the author was reminded, by Andrew Casson, of Milgram's finiteness obstruction computations [40], [41], and, by Ib Madsen, of the related computations of Bentzen and Madsen [4], [36]. The earlier paper of Wall [74] should also be mentioned.

The *ad hoc* calculations which began this study are no longer evident in our exposition, although some vestigial remains can be found in [27]. The key to their systematic treatment was the realization of the fundamental importance of Swan's Isomorphism Theorem (see Chapter 6). Our development