GRACEFULNESS, GROUP SEQUENCINGS AND GRAPH FACTORIZATIONS

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1 Introduction

In this article we survey recent results on the connection between decompositions of complete graphs which admit given automorphism groups, and some combinatorial problems related to these groups. Although these problems are of combinatorial nature, group theorists may find interest in these problems as well.

Most of the results surveyed here are found in [10], [11], [12] and [14] (we do not give proofs in this article, and the interested reader is referred to the above papers). Some of these results extend and generalize previous results on graph labeling (see [5] for a survey) into results related to any finite groups, and some of the results apply known results on group sequencings. Another result applies the Z^*-Theorem of Glaubermann.

We begin with basic definitions. By K_n (respectively, K_n^*) we denote the complete undirected (respectively, directed) graph on n vertices. (Each pair of different vertices in K_n^* is joined by two arcs (directed edges) with opposite directions.)

Definition 1.1 Let n be a natural number and let F be a directed (res., undirected) graph on m vertices, where $m \leq n$. The problem of determining whether or not the edge set of K_n^* (res., K_n) can be partitioned into arc (res., edge) disjoint subgraphs isomorphic to F will be denoted by $P_n^*(F)$ (res., $P_n(F)$). If a solution to this problem exists, we shall say that there exists a *decomposition* of K_n^* (res., K_n) into arc (res., edge) disjoint subgraphs isomorphic to F, and we shall denote this fact $F \mid K_n^*$ (res., $F \mid K_n$). If $m = n$, i.e. each subgraph in the decomposition is a spanning subgraph of K_n^* (res., K_n), we shall say that $F \mid K_n^*$ (res., $F \mid K_n$) is a *factorization* of K_n^* (res., K_n) into factors isomorphic to F. In this case (i.e., when $m = n$) we shall write briefly $P^*(F)$ (res., $P(F)$) instead of $P_n^*(F)$ (res., $P_n(F)$).

Notice that a decomposition problem can always be handled as a factorization problem, by considering a corresponding factor with added isolated vertices.

Definition 1.2 Let F be a directed (res., undirected) graph on n vertices, and let G be a permutation group on the vertices of K_n^* (K_n). We say that K_n^* (K_n) admits a *G-transitive factorization*, into factors isomorphic to F, if there exists a factorization of K_n^* (K_n) such that G acts transitively on the factors in the factorization. If the action of G on these factors is regular, we shall say that the factorization is *G-regular*. The problem of determining whether or not K_n^* (K_n) admits a G-regular factorization into factors isomorphic to F is denoted by $RP^*(G;F)$ ($RP(G;F)$). The problem of determining whether or not K_n^* (K_n) admits *any* regular factorization into factors isomorphic to F is denoted by $RP^*(F)$ ($RP(F)$).

Notice that when F is a directed (undirected) factor of K_n^* (K_n) and $RP^*(G;F)$ ($RP(G;F)$) has a solution, then the order of G must be $n(n-1)/|E(F)|$ (res., $\frac{1}{2}n(n-1)/|E(F)|$), where $E(F)$ denotes the arc (edge) set of F.

This article is organized as follows: In Section 2 we deal with the connection between the problems $RP^*(G;F)$, $RP(G;F)$ and a certain type of labeling of the vertices of F by the elements of G. These labelings generalize and extend well known labeling problems by natural numbers.

In Section 3 we deal with the problem $RP^*(G;F)$, where F is a disjoint union of cycles. In order to treat these problems, we apply and extend the well known concept of *group sequencing*. In the particular case when the factor F consists of a single cycle, known results on group sequencing are applied to obtain corresponding classification theorems.

Section 4 deals with the connection between transitive large sets of disjoint decompositions and group sequencings (see Section 4 for the corresponding definitions). In order to obtain classification results on such large sets, we apply the known concept of R-sequencing of groups. We define also the notion of a Frobenius set, which is a set of permutations having some properties similar to Frobenius groups. We show that the existence of such sets is equivalent to the existence of corresponding large sets.

In Section 5 we study a special case of decompositions, the *inner-transitive Hering decompositions*. We show that the classification of such decompositions is equivalent to the classification of pairs (τ, N), where N is a group and τ is an automorphism of a special type of N, called an (n,t) automorphism (see Section 5 for the corresponding definitions). In order to obtain this equivalence, we use Glaubermann's Z^* Theorem [7] and a generalization of this theorem (proved by Artemovich [3]).

For a directed (or undirected) graph H, we shall use the standard notation $V(H)$ and $E(H)$ for the vertex set and arc (or edge) set of H, respectively. *All groups in this article are finite.* We use the standard notation of group theory.

2 Regular factorization and labeling of the vertices of a graph by group elements

When dealing with regular factorizations of the complete graph K_n^* (Definition 1.2) two fundamental problems arise:

1. For which factors F of K_n^* does there exist a solution to $RP^*(F)$;
2. Classify all pairs (G,F), where G is a group and F is a factor of K_n^*, for which $RP^*(G;F)$ has a solution.

It emerges that the above problems are closely related to a problem of a certain labeling of the vertices of F by the elements of G, the G-*graceful labeling*. The definition is as follows.

Definition 2.1 Let G be a group of order n and let H be a (not necessarily connected) directed graph with n vertices and $n-1$ arcs. We say that H is G-*graceful* if there exists a bijection $f : V(H) \to G$ such that $G - \{1\} = \{f(v)f(u)^{-1} \mid (u, v) \in E(H)\}$.

Notice that in Definition 2.1 all the elements of the multiset $\{f(v)f(u)^{-1} \mid (u, v) \in E(H)\}$ are distinct. Notice also that F may have isolated vertices.

Example 2.2 The following are three simple examples of G-graceful graphs. The corresponding G-graceful labelings are specified by the notation of the vertices.

1. Let G be the additive group $\mathbf{Z}_4 = \{0, 1, 2, 3\}$ and let H be the following directed graph: $V(H) = \{0, 1, 2, 3\}$, $E(H) = \{(1, 0), (1, 2), (1, 3)\}$.
2. Let G be the additive group $\mathbf{Z}_5$ and let H be the following directed graph: $V(H) = \{0, 1, 2, 3, 4\}$, $E(H) = \{(1, 2), (2, 4), (4, 3), (3, 1)\}$ (i.e., H has a unique isolated vertex).
3. Let G be the additive group $\mathbf{Z}_6$ and let H be the following directed graph: $V(H) = \{0, 1, 2, 3, 4, 5\}$, $E(H) = \{(0, 1), (0, 2), (0, 5), (2, 5), (1, 5)\}$ (i.e., H has two isolated vertices).

The following result ([10], Theorem 3.2) clarifies the connection between regular factorization problems and the notion of G-graceful graphs.

Theorem 2.3 *Let G be a group of order n and let F be a directed graph with n vertices and $n-1$ arcs. Then $RP^*(G; F)$ has a solution if and only if F is G-graceful.*

Remark 2.4 When a bijection $f : V(F) \to G$ is given for a G-graceful graph F (like in Definition 2.1), we shall identify each vertex v of F with its image in G, i.e., with $f(v)$. Thus the edges of F are some ordered pairs of elements of G. Moreover, the distinct copies of F in the factorization $F \mid K_n^*$ (whose existence is ensured by Theorem 2.3) are given by group multiplication on the right. More precisely, the distinct copies of F are F^g, $g \in G$, where (x, y) is an arc of F if and only if (xg, yg) is an arc of F^g (here $x, y \in G$).

Theorem 2.3 extends a result of Rosa [16] on β-*valuations* and ρ-*valuations*. These notions are defined as follows.

Definition 2.5 Let H be an undirected graph.

1. An injection $f : V(H) \to \{0, 1, \ldots, |E(H)|\}$ is called a β-*valuation* (*graceful labeling*) if $|f(x) - f(y)|$ are distinct for all $\{x, y\} \in E(H)$. A graph which admits a β-valuation is called a *graceful* graph.
2. An injection $f : V(H) \to \{0, 1, \ldots, 2|E(H)|\}$ is called a ρ-*valuation* if $\{|f(x) - f(y)| \mid \{x, y\} \in E(H)\} = \{a_1, a_2, \ldots, a_{|E(H)|}\}$, where $a_i = i$ or $a_i = 2|E(H)| + 1 - i$.

Rosa introduced the above notions as a tool for decomposing the undirected complete graph K_{2n+1} into isomorphic subgraphs on $n+1$ vertices. In particular, these notions were a tool for attacking the well known conjecture of Ringel [15] which says:

Conjecture. Let T be any tree on $n+1$ vertices. Then $T \mid K_{2n+1}$.

Rosa proved that a tree T on $n+1$ vertices has a ρ-valuation if and only if Ringel's conjecture holds for T. Therefore, if T has a β-valuation then $T \mid K_{2n+1}$. Since Rosa published his results [16], problems on graph valuations attracted a lot of attention. An updated survey on this subject, including many references, is found in [5].

It can be shown that the above results of Rosa are, in a sense, particular cases of Theorem 2.3. For the result on β-valuation, the reader is referred to [10], Corollary 4.3. In order to see how the result on ρ-valuation follows from Theorem 2.3, we shall confine ourselves with the following example.

Example 2.6 Let T be the following undirected tree:

$$V(T) = \{v_0, v_1, v_2, v_3, v_4\},$$

$$E(T) = \{\{v_0, v_1\}, \{v_0, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}\}.$$

Notice that T has 4 edges. We shall obtain by Theorem 2.3 that $T \mid K_9$ if and only if T has a ρ-valuation. Let $G = \mathbf{Z}_9$, and let T^* be the directed factor of K_9^* obtained from T by adding 4 isolated vertices and considering the directed graph obtained when each edge of T is replaced by two opposite arcs. Notice that T^* has 9 vertices and 8 arcs. Consider the following $\mathbf{Z}_9$-graceful labeling of T^*:

$$\begin{aligned} V(T^*) &= \{0, 1, 2, \ldots, 8\}, \\ E(T^*) &= \{(0,1), (1,0), (0,2), (2,0), (2,5), (5,2), (2,6), (6,2)\}. \end{aligned}$$

By Theorem 2.3 $T^* \mid K_9^*$, and so it easily follows that $T \mid K_9$. Furthermore, one easily verifies that the given $\mathbf{Z}_9$-graceful labeling of T^* yields a corresponding ρ-valuation of T. More generally, as in the current example, Theorem 2.3 implies that for a tree T with $n+1$ vertices, $T \mid K_{2n+1}$ if and only if T has a ρ-valuation.

We note that Theorem 2.3 extends also another well known problem, namely the using of difference sets for the construction of block designs. The reader is referred to [10], Section 4.

We conclude this section with the following theorem, which enables us to treat factorizations $F^* \mid K_n^*$ (or $F \mid K_n$), when F is an arc (edge) disjoint union of cycles. Such factorizations and their connection to group sequencing problems, will be treated in the next section.

Theorem 2.7 *Let F be a directed graph such that $|V(F)| = |E(F)| = n$, and let G be a group of order $n-1$. Then $RP^*(G;F)$ has a solution if and only if there exists a vertex $v_0 \in V(F)$ of outdegree 1 and indegree 1, such that $F - \{v_0\}$ is G-graceful.*

3 Regular Oberwolfach problems and group sequencings

We begin with the following definitions.

Definition 3.1 An *Oberwolfach factorization* of K_n^* (res., K_n) is a factorization of K_n^* (K_n) into copies of a graph F, where F is an arc (edge) disjoint union of directed (undirected) cycles.

Definition 3.2 Let G be a permutation group on $V(K_n^*)$ ($V(K_n)$). The problem of determining whether there exists a G-regular Oberwolfach factorization of K_n^* (K_n) into copies isomorphic to F will be denoted $ROP^*(G;F)$ ($ROP(G;F)$). (Recall the definition of regular factorizations given in Definition 1.2.)

The Oberwolfach factorization problem of K_n^* (or K_n) for a given F is far from being settled. See [1] for a survey on this subject. When F consists of a single cycle, then F is a hamiltonian cycle, and the corresponding factorization is called a *hamiltonian factorization*. The existence of a hamiltonian factorization for the undirected case (where n must be odd) is well known. The existence of a hamiltonian factorization for the directed case, for every n, was established by Tilson [17].

The notion of G-regular hamiltonian factorizations is closely related to the well known notion of group sequencing (see Theorem 3.4 below). We give first the corresponding definition.

Definition 3.3 Let G be a group of order n. A *sequencing* of G is a sequence $a_1, a_2, \ldots, a_n$ of all the distinct elements of G, such that all the partial products $a_1, a_1a_2, \ldots, a_1a_2\cdots a_n$ are distinct (note that necessarily $a_1 = 1$). A group having a sequencing is called a *sequenceable* group.

The classification of all the sequenceable groups is a well known problem, which is still not settled. However, various infinite families of sequenceable groups are known. It is conjectured that all the non-abelian groups of order larger than 8 are sequenceable (see [13], Section 5.4 for further details).

Applying Theorem 2.7, the following result can be proved (see [11], Corollary 3.1).

Theorem 3.4 *Let $n \geq 3$ and let G be a group of order $n-1$. Then K_n^* admits a G-regular hamiltonian factorization if and only if G is sequenceable.*

Using known results on group sequencing, we obtain (see [11], Corollary 3.2):

Corollary 3.5 *Let $n \geq 3$ and let G be a group of order $n-1$. Then K_n^* admits a G-regular hamiltonian factorization if one of the following cases holds:*

1. *G is a solvable group with a unique element of order 2, except the case when G is the quaternion group Q_4;*
2. *G is a dihedral group, except the cases D_3 and D_4;*
3. *G is a dicyclic group, except the case when G is the quaternion group Q_4;*

4. G is a non-abelian group of order pq, where $p < q$ are odd primes such that p has 2 as a primitive root.

We turn now to G-regular hamiltonian factorizations of the undirected graph K_n. Notice first that in this case n must be odd. Applying known results on group sequencing, we obtain a full classification, as specified in the following theorem.

Theorem 3.6 *Let $n \geq 3$ be an odd integer and let G be a group of order $(n-1)/2$. Then K_n admits a G-regular hamiltonian factorization if and only if $n \equiv 3 \pmod 4$.*

For the proof of this theorem we applied Proposition 3.8 below. We include first the definition of the known concept of *symmetric* sequencing.

Definition 3.7 Let G be a group of order $2m$ with a unique element of order 2, say z. A *symmetric sequencing* of G is a sequencing $a_0, a_1, \ldots, a_{2m-1}$ of G such that $a_m = z$ and $a_{m+i} = a_{m-i}^{-1}$ for $1 \leq i \leq m-1$ (notice that the first element is denoted by a_0 instead of a_1).

Proposition 3.8 *Let $n \geq 3$ be an odd integer and let G be a group of order $(n-1)/2$. Let $G_1 = \langle z \rangle \times G$, where z is an element of order 2, and suppose that G_1 has a symmetric sequencing. Then K_n admits a G-regular hamiltonian factorization.*

Proposition 3.8 follows by Lemma 4.4 and Theorem 5.2 in [11]. For the proof of Theorem 3.6 we applied, besides Proposition 3.8, a result of Anderson and Ihrig [2], which asserts that every finite solvable group which has a unique element of order 2, except the quaternion group Q_4, has a symmetric sequencing. For the detailed proof (which uses also the theorem of Feit and Thompson) the reader is referred to [11], Sections 4 and 5.

The general case of G-regular Oberwolfach factorizations was also treated in [11]. It was shown there that the existence of G-regular Oberwolfach factorizations is equivalent to the existence of some sequencings (called $(l_1, \ldots, l_t)$-sequencings) of the group G, which are extensions of the original notion of sequencing. These sequencings may be used to obtain new G-regular factorizations. We shall not discuss further these results, and the reader is referred to [11] for details.

4 Transitive large sets of disjoint decompositions and group sequencings

The notions of *large sets* and *transitive large sets* are defined as follows.

Definition 4.1 A *large set of disjoint decompositions* of K_n^* (K_n) into cycles of length k, denoted by k-LSD, is a partition of the set of all cycles of length k in K_n^* (K_n) into disjoint decompositions of K_n^* (K_n) (i.e., no two decompositions have a common cycle). We shall say that such a k-LSD is *transitive*, or *H-transitive*, if there exists a permutation group H on the vertices of K_n^* (K_n) such that H is transitive on the decompositions in the k-LSD. A transitive k-LSD will be denoted by k-TLSD.

Notice that an n-LSD for K_n^* (or K_n) is a large set of disjoint *hamiltonian factorizations*. This subject was treated by Bryant in [4]. In particular, it was proved there that for each odd $n \geq 3$ there exists a large set of disjoint hamiltonian factorizations of K_n.

Theorem 3.4 on G-regular hamiltonian factorizations enables us to obtain a result on an n-TLSD for K_n^* (Theorem 4.3 below). However, we need first the following definition.

Definition 4.2 Let G be a group. If one (and so, up to group isomorphism, all) of the decompositions in a k-TLSD is G-regular, we shall say that the k-TLSD is *G-regular*.

It emerges that when a regular hamiltonian factorization of K_n^* is given, or, equivalently (see Theorem 3.4), when a sequenceable group of order $n-1$ is given, we can construct a corresponding n-TLSD for K_n^*. Thus we have (see [12], Theorem 1):

Theorem 4.3 *Let G be a group of order $n-1$, where $n \geq 3$. Then there exists a G-regular n-TLSD for K_n^* if and only if G is sequenceable.*

Applying known results on group sequencing, we obtain (see [12], Corollary 1.1):

Corollary 4.4 *There exists a regular n-TLSD for the following values of n:*
1. All odd n, $n \geq 3$;
2. All n such that $n = pq + 1$, where $p < q$ are odd primes and p has 2 as a primitive root.

For the undirected case, we have (see [12], Theorem 2):

Theorem 4.5 *Let $n \geq 3$ be an odd integer and let G be a group of order $(n-1)/2$. Then there exists a G-regular n-TLSD for K_n if and only if $n \equiv 3 \pmod 4$.*

For each odd n, $n \geq 3$, there exists a group G of order $n-1$ having a symmetric sequencing (see Definition 3.7 and [2]; we can choose, for instance, a cyclic group). By using this we can construct an n-TLSD for the undirected graph K_n. Thus we obtain the following (see [12], Theorem 3).

Theorem 4.6 *For each odd n, $n \geq 3$, there exists an n-TLSD for K_n.*

In [12] we have studied also $n-1$-LSDs, i.e. LSDs of decompositions into cycles of size $n-1$ of K_n^* or K_n. Such cycles were called *almost hamiltonian cycles*, and the corresponding decompositions were called *almost hamiltonian decompositions*. For introducing the corresponding results, we include first the known concept of R-sequencing.

Definition 4.7 Let G be a group of order n. An *R-sequencing* of G is a sequence $a_0 = 1, a_1, \ldots, a_{n-1}$ of all the distinct elements of G, such that $a_0 a_1 \cdots a_{n-1} = 1$ and such that all the partial products $a_0, a_0 a_1, \ldots, a_0 a_1 \cdots a_{n-2}$ are distinct. (Exactly one element of G does not occur among these partial products.) A group having an R-sequencing is called an *R-sequenceable group*.

It emerges that the connection between R-sequenceable groups and regular almost hamiltonian decompositions is similar to the connection between sequenceable groups and regular hamiltonian decompositions. We have (see [12], Theorem 4):

Theorem 4.8 *Let $n \geq 4$ be an integer and let G be a group of order n. Then K_n^* has a G-regular almost hamiltonian decomposition if and only if G is R-sequenceable.*

As for almost hamiltonian regular large sets, we have the following result ([12], Theorem 5).

Theorem 4.9 *Let n be a prime power. Then there exists a regular $(n-1)$-TLSD for K_n^*.*

The proof of Theorem 4.9 is based on a construction which is induced by a Frobenius group of order $n(n-1)$, where n is a prime power. It emerges that the concept of a *Frobenius set* defined below is closely related to regular $(n-1)$-TLSDs of K_n^*.

Definition 4.10 Let G be a regular subgroup of the symmetric group S_n and let H be a subgroup of S_n of order $n-1$. We shall say that the set $HG = \{hg \mid h \in H, g \in G\}$ is a *Frobenius set* if every $x \in HG - \{1\}$ fixes at most one letter from $\{1, 2, \ldots, n\}$. We shall call the groups G, H a *kernel* and a *complement* of the Frobenius set, respectively.

Notice that every Frobenius group of order $n(n-1)$, where n is a prime power, is a Frobenius set.

We define the following.

Definition 4.11 An $(n-1)$-LSD of K_n^* is *strongly S_{n-2}-transitive* if for every pair of different vertices $v_1, v_2 \in V = V(K_n^*)$, the subgroup of $Sym(V)$ of all permutations which fix both v_1 and v_2 acts transitively (and so regularly) on the $(n-2)!$ decompositions in the LSD.

The following result makes clear the connection between the last two definitions. (See Lemma 5.1 in [12].)

Proposition 4.12 *Let G be a group of order n. Then K_n^* admits a G-regular strongly S_{n-2}-transitive $(n-1)$-LSD if and only if there exists a G-regular almost hamiltonian decomposition of K_n^* such that $\langle h \rangle G$ is a Frobenius set, where h is the permutation induced by one of the cycles in the decomposition.*

We conclude this section with the following two conjectures.

Conjecture. A Frobenius set with a kernel G of order n and a cyclic complement H of order $n-1$ exists if and only if n is a prime power.

Clearly, the validity of this conjecture will imply:

Conjecture. There exists a G-regular strongly S_{n-2}-transitive $(n-1)$-LSD of K_n^* if and only if n is a prime power.

5 Inner transitive Hering configurations

In this section we deal with inner transitive factorizations of K_n^* [14].

Definition 5.1 Let $F \mid K_n^*$ be a factorization of K_n^*, where F is a disjoint union of directed cycles and (possibly) some isolated vertices. Denote by $F_1, F_2, \ldots, F_m$ all the (arc disjoint) factors in the factorization. For each factor F_i, $1 \leq i \leq m$, let $\sigma_i \in S_n$ be the permutation defined by $\sigma_i(k) = l$ if and only if (k, l) is an arc of F_i. Let G be the group generated by $\sigma_1, \ldots, \sigma_m$. Then the factorization is *inner transitive* if G permutes the factors $F_1, \ldots, F_m$ transitively.

In many cases the group generated by the σ_is will be S_n or A_n, and the factorization will not be inner transitive. It may be interesting, and certainly not easy, to classify the inner transitive factorizations.

In this section we classify the inner transitive Hering configurations. A Hering configuration is a special type of factorization of K_n^* defined as follows.

Definition 5.2 A factorization $F \mid K_n^*$ is a *Hering configuration* of type t and order n if the following conditioned hold:

1. The factor F is a disjoint union of $(n-1)/t$ directed cycles of size t and one isolated vertex;

2. any two factors in the factorization have exactly one edge (undirected arc) in common. That is, if the common edge is $\{i, j\}$, then the arc (i, j) lies in one factor, and the arc (j, i) lies in the second factor.

The Hering configuration was introduced by Hering [9]. Since then, various papers have been published (see [14], p. 380), which deal mainly with the problem of classifying the pairs (n, t) for which a Hering configuration of type t and order n exists, and with the classification of such configurations. In [14] a classification was given for all the pairs (n, t) for which an *inner transitive Hering configuration* of type t and order n exists. This classification used a deep result in group theory - Glaubermann's Z^*-theorem [6] and its generalization proved by Artemovich [3]. We note that the result of Artemovich uses the classification of the finite simple groups.

A key notion in the classification of inner transitive Hering configurations is the notion of (n, t)-automorphisms, defined as follows.

Definition 5.3 Let N be a finite group. An automorphism τ of N is an (n, t)-automorphism of N if the following conditions hold:

1. $o(\tau) = t$;

2. $[N : C_N(\tau^i)] = n$ for every $1 \le i \le t-1$;
3. $(t, |N|) = 1$;
4. Denote $G = \langle\tau\rangle N$, and let K be the conjugacy class of τ in G. Then for any distinct $x, y \in K$, xy^{-1} does not commute with any element of K.

It emerges that every (n,t)-automorphism induces an inner transitive Hering configuration of type t and order n. We have ([14], Theorem 1):

Theorem 5.4 *Let τ be an (n,t)-automorphism of a group N, where $t \ge 3$, and let $K = \{\tau_1, \tau_2, \ldots, \tau_n\}$ be the conjugacy class of τ in the semidirect product $G = \langle\tau\rangle N$. For each i, $1 \le i \le n$, let F_i be the factor of K_n^* defined as follows: for $v_k, v_l \in V(K_n^*)$, $(v_k, v_l) \in E(K_n^*)$ if and only if $\tau_i^{-1}\tau_k\tau_i = \tau_l$. Then the F_is provide an inner transitive Hering configuration of type t and order n (with corresponding transitive group $G = \langle\tau\rangle N$).*

Notice that Theorem 5.4 enables us to construct various Hering configurations. Furthermore, using the Z^*-theorem (and its generalization) we can show that *all* the inner transitive Hering configurations can be constructed in the way described in 5.4. More precisely, we have the following result ([14], Theorem 3).

Theorem 5.5 *Let $t \ge 3$ be a natural number. Then any inner transitive Hering configuration of type t and order n is induced (in the sense of 5.4) by some (n,t)-automorphism.*

The above results enables us to classify all the pairs (n,t) for which an inner transitive hering configuration of type t and order n exists (these are exactly the pairs (n,t) for which an (n,t)-automorphism exists). We conclude our brief survey with the following (see Theorem 5 in [14]).

Theorem 5.6 *Let $t \ge 3$ be a natural number and let $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$, where the p_is are all the distinct prime divisors of n. Then there exists an inner transitive Hering configuration of type t and order n if and only if t divides $p_i^{e_i} - 1$ for every $1 \le i \le r$.*

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